Jim Gatheral's 60th Birthday Conference

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Modelling Long Term Capital Market Risks With Stochastic Optimization

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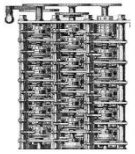
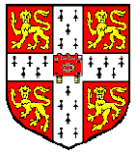
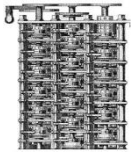




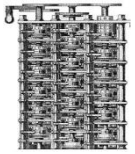
Outline

- Introduction
- Stochastic optimization technology
- Robust long term yield curve modelling
- Pricing variable annuity products
- Unit-linked guaranteed product fund management
- Conclusion



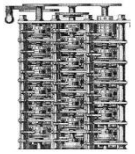
Dynamic Stochastic System Optimization

- There are four theoretical/computational approaches to the optimization of Markovian stochastic dynamical systems
  - Discrete state and time dynamic programming using Bellman's principle of optimality and forward or backward recursion or policy iteration
  - Discrete state and time Markov chains using linear programming techniques pioneered by Howard
  - Continuous state and time dynamic programming solving the Bellman PDE numerically
  - Dynamic stochastic programming in discrete time using mathematical programming algorithms
- Of these only dynamic stochastic programming can handle an arbitrary number of risk factors – the others are restricted to 3 or 4 – and DSP can relax the Markovian assumption practically



Dynamic Stochastic Programming

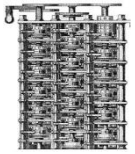
- Dynamic stochastic programming is a means of solving dynamic stochastic optimization problems where future uncertainty is given by a large number of random processes and decisions have specified future timings
  - General idea of dynamic stochastic programming
- Model future decisions as well as current ones to give a complete forward plan to the planning horizon
- Incorporate many alternative futures in the form of simulated scenarios of the underlying risk factors against which decisions are robust
- Optimize all decisions simultaneously



Financial Optimization Applications

Experience with a variety of actual applications including

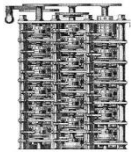
- Equity and credit trading hedge fund strategies
- Long term asset allocation
- Asset liability management
- Derivative portfolio pricing and hedging strategies
- Risk management
- Capital allocation
- Real options evaluation
- Financially hedged logistics operations



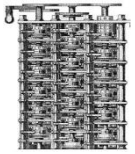
Some Current ALM Applications of DSP

- **Pioneer Investments** – guaranteed return products & DB pension schemes in the EU
- **Allianz** – property and casualty insurance globally
- **Siemens** – DB pension schemes in Germany & Austria
- **Aon-Hewitt** – DC pension schemes in the EU
- **Freddie Mac** – mortgage pool funding in the US

and many more ...



Stochastic Optimization Technology



Dynamic Stochastic Programme

- Consider a financial planning problem formulated as a canonical linearly constrained dynamic stochastic programming (DSP) problem in recourse form (boldface denotes random entities)

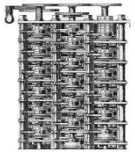
$$\max_{x_1}\{\, f_1(x_1) + E_{\omega_2}[\max_{x_2}(\mathbf{f}_2(\mathbf{x}_2) + \ldots + E_{\omega_T|\omega^{T-1}}[\max_{\mathbf{x}_T} \mathbf{f}_T(\mathbf{x}_T)])] \quad \}$$

$$\begin{aligned}
\text{s.t.} \quad & A_1 x_1 & & & & & & = & b_1 & \\
& \mathbf{A}_{21} x_1 & + & \mathbf{A}_{22}\mathbf{x}_2 & & & & = & \mathbf{b}_2 & \text{a.s.} \\
& & & \mathbf{A}_{32}\mathbf{x}_2 & + & \mathbf{A}_{33}\mathbf{x}_3 & & = & \mathbf{b}_3 & \text{a.s.} \\
& & & & & \ldots & & : & & \\
& & & & & & \mathbf{A}_{TT}\mathbf{x}_T & = & \mathbf{b}_{T+1} & \text{a.s.}
\end{aligned}$$

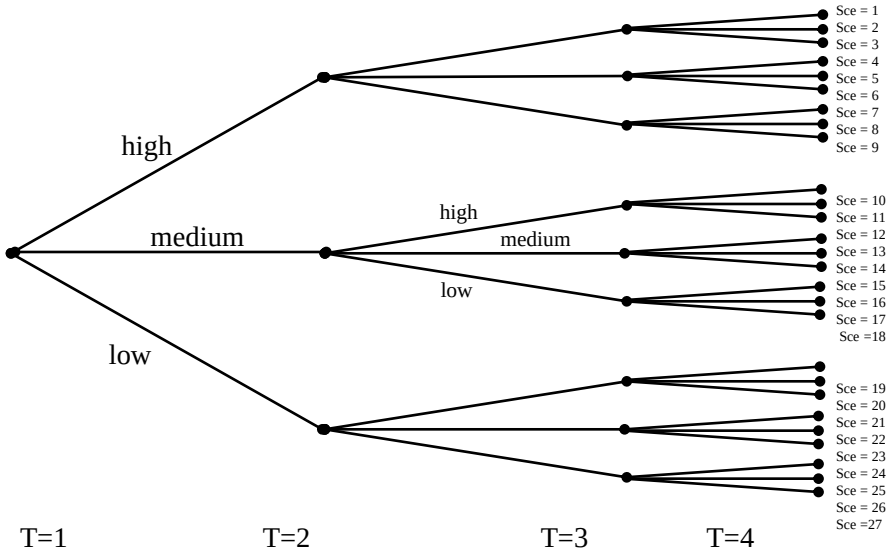
$$l_1 \le x_1 \le u_1$$

$$l_t \le \mathbf{x}_t \le u_t \quad a.s. \quad t = 2, \ldots, T+1$$

- $\boldsymbol{\omega}_t := \{\mathbf{b}_t, \mathbf{f}_t, \mathbf{A}_{t-1,t}, \mathbf{A}_{tt}\} \quad for \quad t = 2, \ldots, T+1$
- $E_{\omega_T|\omega^{T-1}}$ denotes conditional expectation wrt the **history** $\boldsymbol{\omega}^{T-1}$ of the data process $\boldsymbol{\omega}$



Scenario Tree

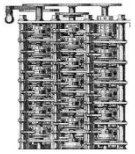


Multi-level Scenario Tree

Time period : decision stage of the problem

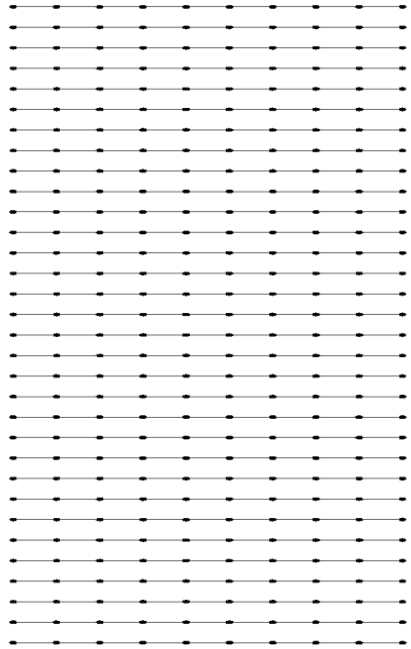
Each node : random event ω_t conditioned on the past realization ω^{t-1}

- **After taking action at stage t under uncertainty due to branching scenarios representing the realized values of the r.v. s make corrective actions (recourse) at the next stage of the tree**
- **This structural schema requires vector process simulation of branches conditionally from each node of the tree**
- **Tree size in terms of nodes increases exponentially in the number of stages and linearly in the number of scenarios**

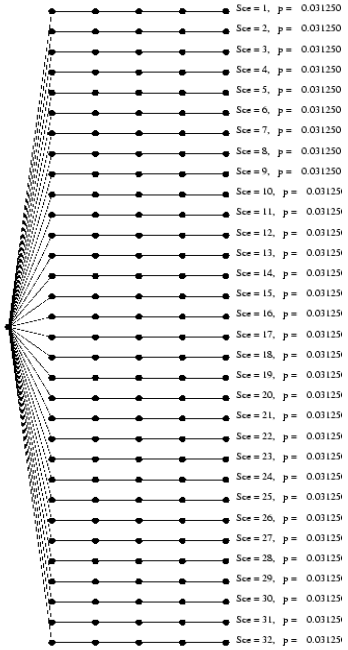


Scenario Generation

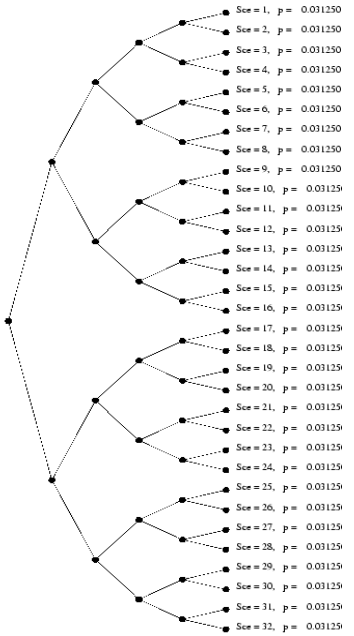
Alternative representations of possible futures



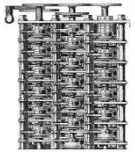
Distribution Problem/DFA



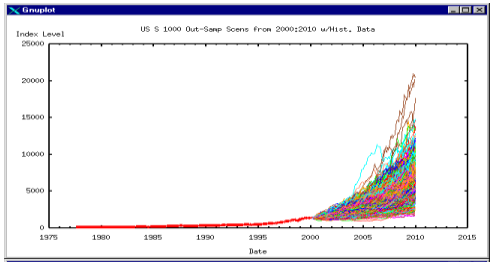
2 Stage Problem

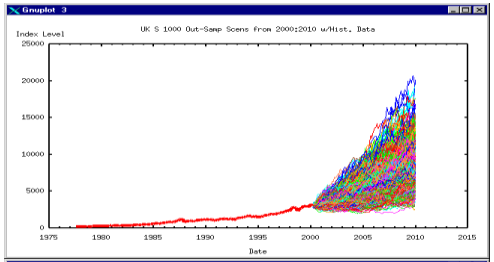


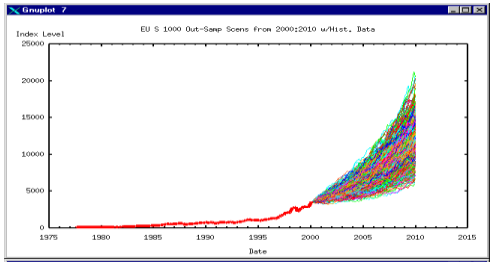
Multistage Problem

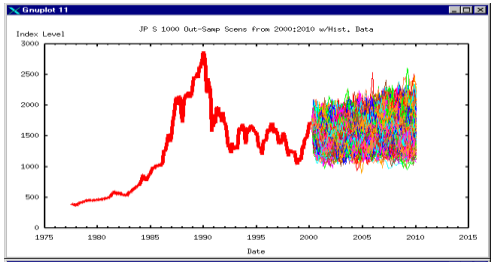


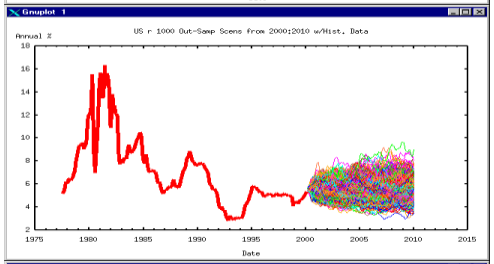
Ten year out-of-sample scenario forecasts to 2010 1977-2000

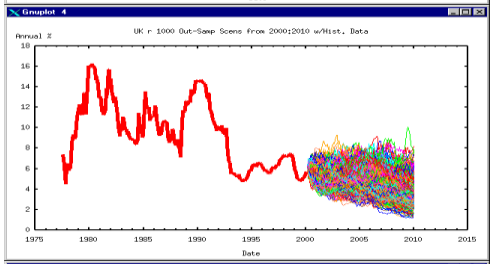


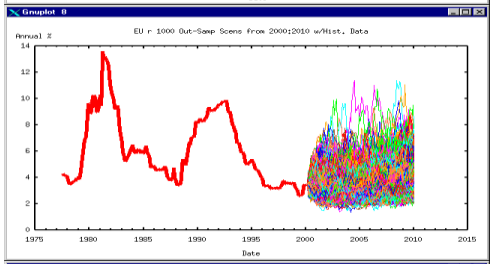


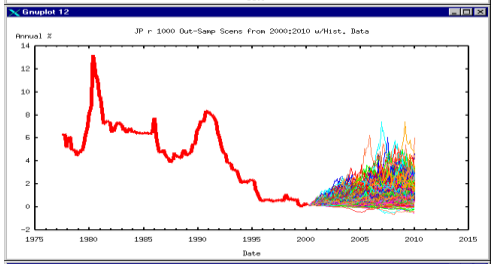


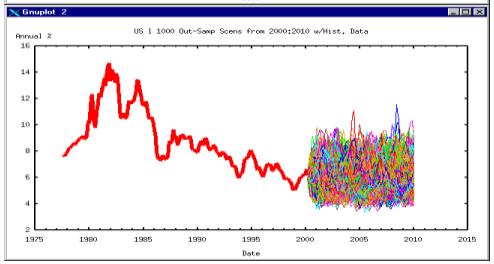


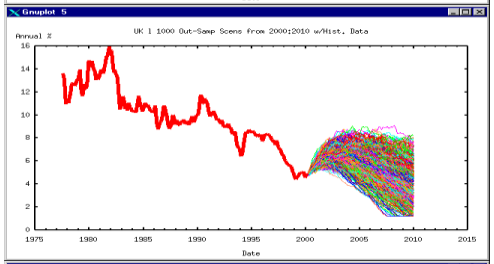


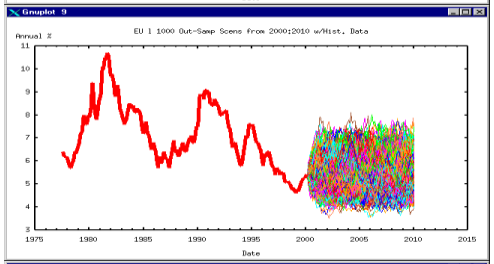


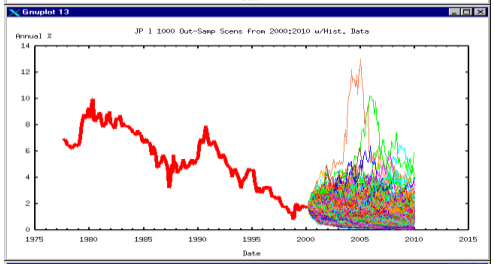


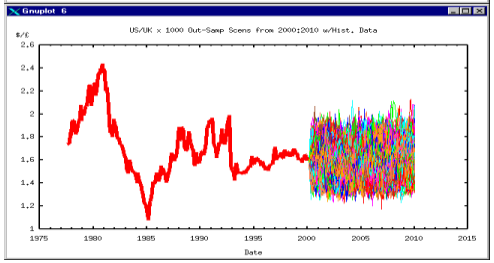


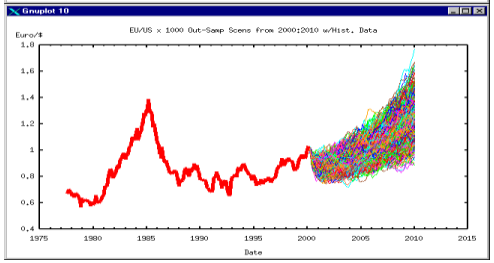


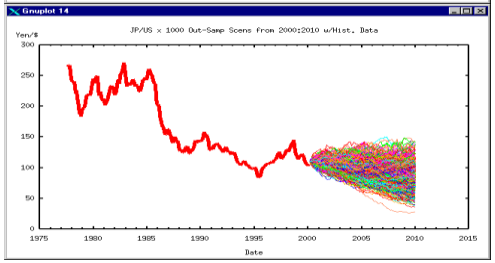


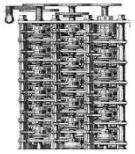




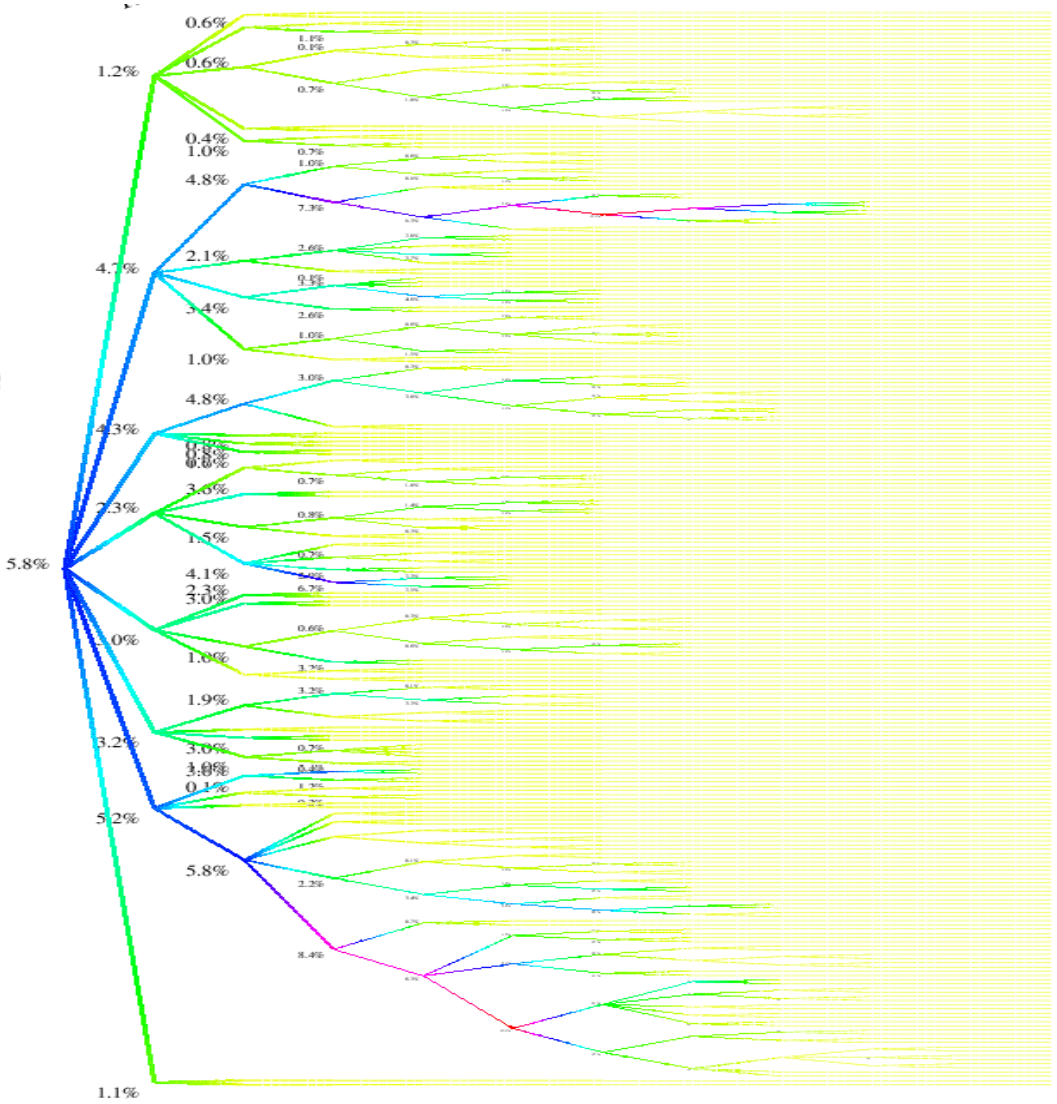


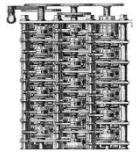


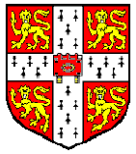




Dynamically Sampled Scenario Tree







Deterministic Equivalent

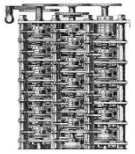
This leads to a large LP in the linear case where $\omega^t \in W$ is a possible realization of the random vector $\boldsymbol{\omega}^t$ and corresponds to a node of the scenario (data path) tree Dantzig & Madansky (1960)

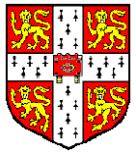
$$\pi := \Big\{ \min_{x_1} \; c_1 x_1 + \sum_{\omega^2 \in \Omega^2} p_2(\omega^2) c_2(\omega^2) x_2(\omega^2) + \sum_{\omega^3 \in \Omega^3} p_3(\omega^3) c_3(\omega^3) x_3(\omega^3)] \\ \ldots + \sum_{\omega^T \in \Omega^T} p_T(\omega^T) c_T(\omega^T) x_T(\omega^T) \Big\}$$

$$\begin{array}{llllllllll} \text{s.t.} & A_{11}x_1 & & & & & & = & b_1 & \\ & A_{21}(\omega^2)x_1 & + & A_{22}(\omega^2)x_2(\omega^2) & & & & = & b_2(\omega^2) & \text{a.s.} \\ & A_{31}(\omega^3)x_1 & + & A_{32}(\omega^3)x_2(\omega^2) & + & A_{33}(\omega^3)x_3(\omega^3) & & = & b_3(\omega^3) & \text{a.s.} \\ & & : & & : & & \ldots & : & & \\ & A_{T1}(\omega^T)x_1 & + & A_{T2}(\omega^T)x_2(\omega^2) & + & \ldots & + A_{TT}(\omega^T)x_T(\omega^T) & = & b_T(\omega^T) & \text{a.s.} \end{array}$$

$$l_1 \le x_1 \le u_1$$
$$l_t \le x_t(\omega^t) \le u_t$$
$$\omega^t \in \Omega^t \qquad t = 1, \ldots, T$$

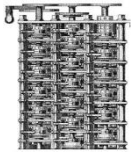
Matrix size increases exponentially with the number of time stages and linearly with the number of scenarios





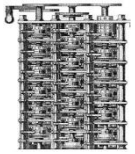
Market and Pricing Measures

- In asset liability management using DSP for liability driven investment (LDI) solutions requires the incorporation of long dated interest rate and inflation swaps in the models
- The technical problem in incorporating such instruments into portfolio construction is the consistency of the data used to price them with that used to generate traditional instrument expected returns since the former requires the risk-neutral (risk-discounted) or pricing probability measure Q while the latter requires the market (real-world) measure P



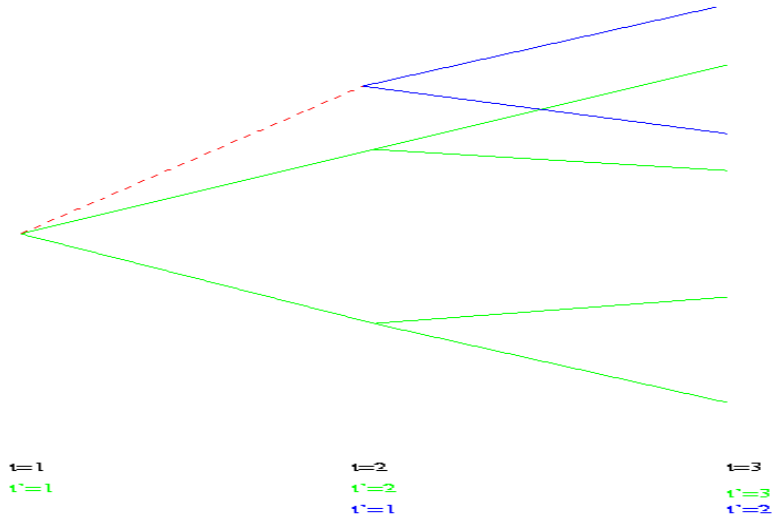
Problem Generation and Solution Methods

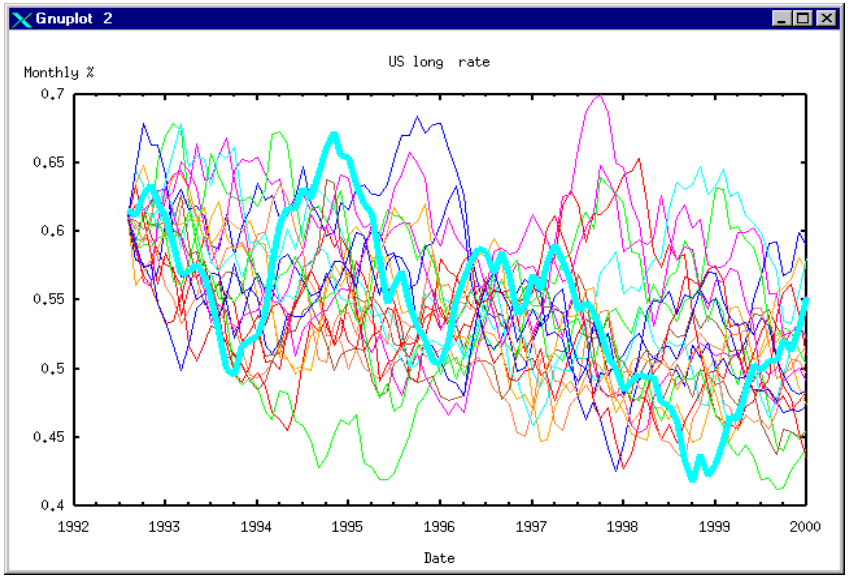
- Deterministic equivalent of the stochastic program (SP) is convex but possibly nonlinear
- Approximation – very large sparse linear programming (LP) problem
- Solution method depends on utility/risk function
  - Downside-Quadratic: nested Benders or CPLEX barrier interior point
  - Exponential, Power/Log: nested Benders
  - Linear: nested Benders or CPLEX barrier



Implementation

- Implement only the first period decision (portfolio)
- This *implementable decision* is robust against alternative scenarios including extremes
- Underlying dynamic economic scenario generator is updated at each portfolio rebalance





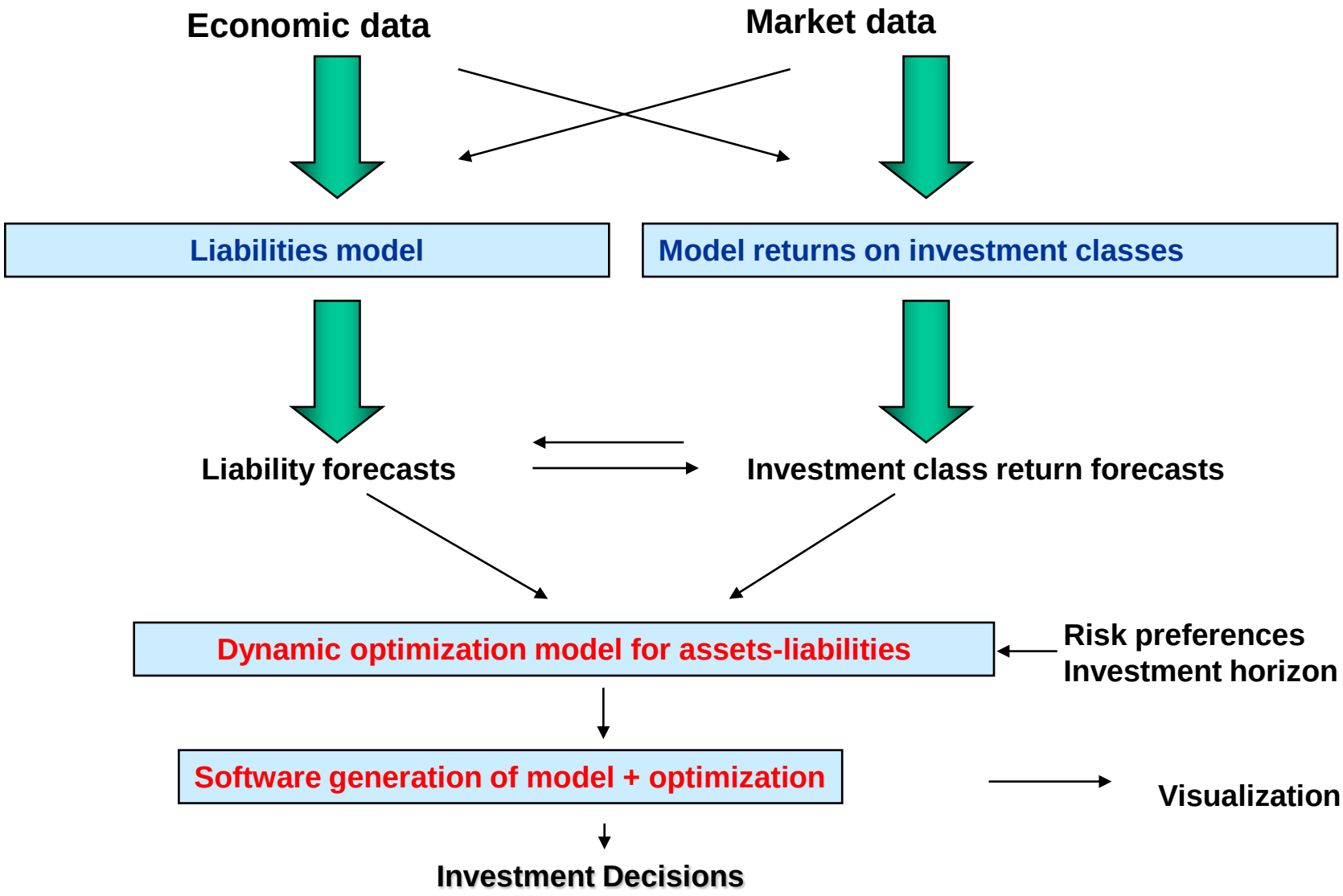
Strategic Financial Planning

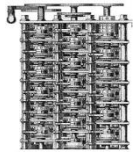
Gather Data
Statistical Analysis of Data

Econometric Modelling

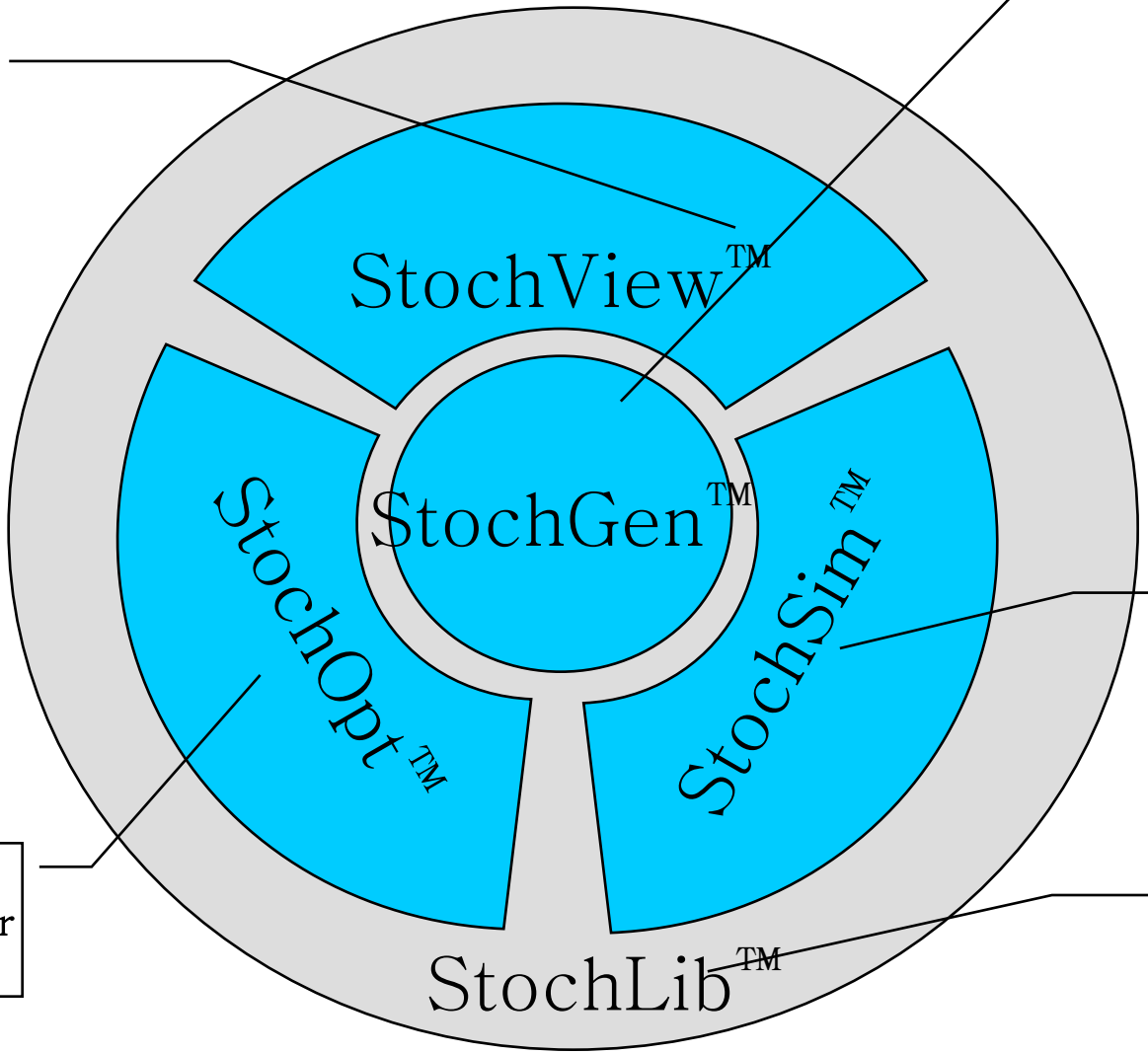
Monte Carlo Simulation

Optimization Model and Fund Objectives and Constraints





Powered by Stochastics™



Visualisation of data, problem & results (Java library)

Dynamic stochastic programme generator (DSP modelling language GSPL)

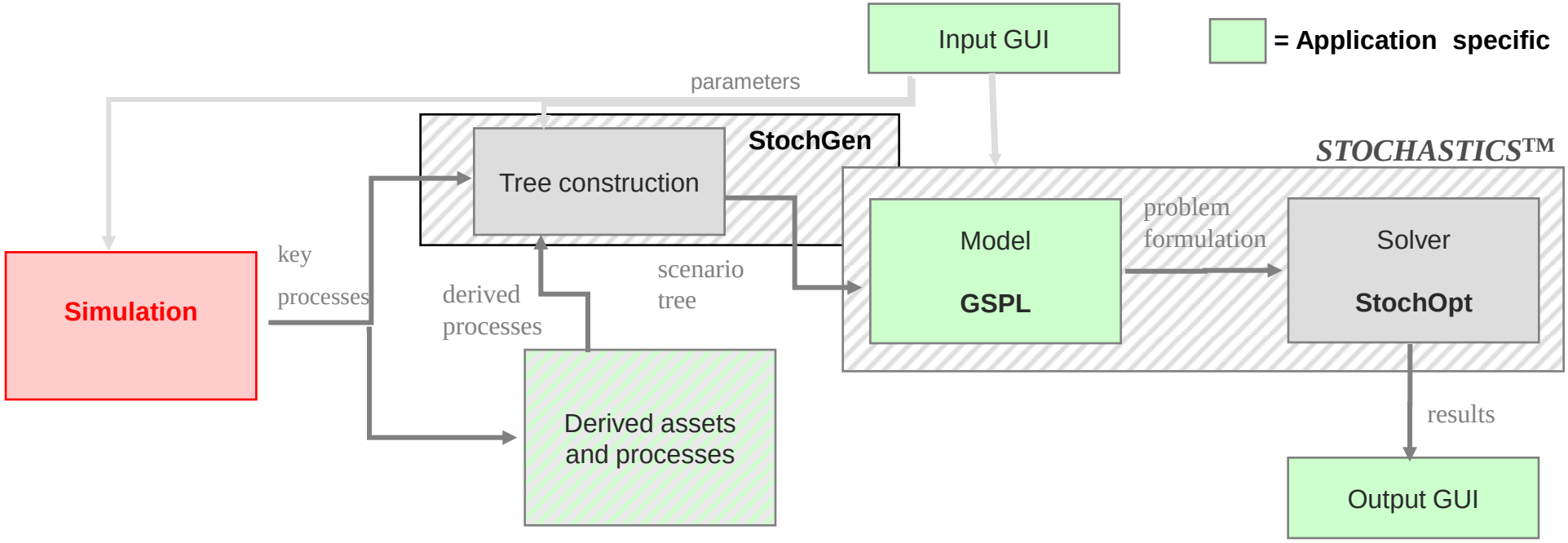
Simulation of tree of future scenarios (library)

Dynamic stochastic programme optimizer (DSP solver GNBS)

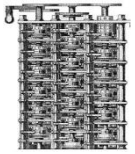
Library of Stochastics™ components

Fundamentals of the *Stochastics*™ System

- Simulation is crucial in the optimization process but
  - difficult and complex for any application
  - a separate problem to model building
  - needs to concentrate only on key processes (others can be derived)

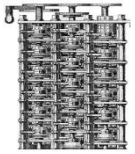


Robust Long Term Yield Curve Modelling



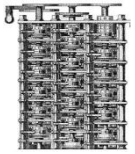
Yield Curve Model Applications

- Scenario simulation for predominantly long term asset liability management (ALM) problems in multiple currencies
- Valuation of complex structured derivatives and other products and portfolios with embedded derivatives in multiple currencies
- Risk assessment of portfolios and structured products



Variety of Approaches to Yield Curve Modelling

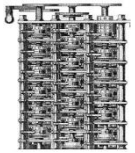
- Investment bank pricing and hedging of fixed income products
  - Short term current market data calibration
  - Updated for re-hedging
  - Evaluated by *realized hedging P&L*
- Central bank forecasts for monetary policy making
  - Long term historical estimation for medium term forecasting
  - Updated for next forecast
  - Mainly evaluated by *in-sample fit to historical data*
- Consultants and fund managers advice for product pricing, investment advice and asset liability management over long horizons
  - Long term historical calibration to market data often using filtering techniques
  - Updated for decision points
  - Evaluated by *consistency with out– of –sample market data*: e. g. prices, returns



Model Requirements

- Continuous time
- Mean reversion
- Dynamic evolution under both pricing (risk neutral) *and* market (real world) measures
- Wide range of yield curve shapes and dynamics reproduced (LIBOR)
- Realistic zero lower bound (ZLB) modelling
- Feasible and efficient discount bond price or yield calculation
- Parameter estimation by efficient model calibration to market data to multiple yield curves and currency exchange rates
- Parsimony in parameter specification
- Time homogeneity

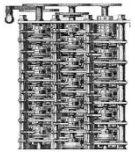
Dempster *et al.* (2010, 2014, 2015)



Multi-factor Yield Curve Models

- Three broad overlapping classes
  - Short rate models
  - Heath-Jarrow-Morton models
  - Market models
- Most rate variability captured by 3 stochastic factors

  Litterman & Scheinkman (1991)
- The 2 factor affine or quadratic short rate models are insufficient to reproduce the correlation structure of market rate changes but 3 to 5 factors suffice
  Rebonato & Cooper (1995) Nawalka & Rebonato (2011)
- The Nelson-Siegel (1987) 3-factor short rate model widely used by central banks has time inhomogeneous parameters and is neither parsimonious nor arbitrage free
- The Diebold-Rudebusch (2011, 2013) version of this model corrects both these faults
  Rebonato (2015)



3 Factor Affine Short Rate Models

- The 3 factors under the pricing (risk-neutral) measure Q satisfy the $A_0(3)$ SDE

$$d\boldsymbol{Y}_t = \Lambda(\Theta - Y_t)dt + \Sigma\sqrt{S_t}\,d\boldsymbol{W}_t$$

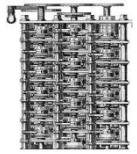
$$[S_t]_{ii} = \alpha_i + \beta_i' Y_t \qquad i = 1,2,3$$

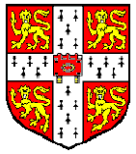
- Discount bond prices are given in affine form as $P_t(\tau) = e^{A(\tau)+B(\tau)'Y_t}$
  and the instantaneous short rate similarly as $r_t = \phi_0 + \phi_x' Y_t$

- Then bond prices and yields are given respectively by $P_t(\tau) = E^Q\left[\exp\left(-\int_t^{t+\tau} \boldsymbol{r}_s ds\right)\right]$
  and $y_t = -\ln P_t(\tau)/\tau$

- A 3 rate vector satisfies the Ricatti equation $\dfrac{\partial R_t(\tau)}{\partial \tau} = \Lambda R_t(\tau) - \dfrac{1}{2} R_t(\tau)\Sigma S \Sigma' R_t(\tau)' + r_t 1$

Duffie & Kan (1996) Dai & Singleton (2000) Dempster *et al.* (2014)





3 Factor Gaussian Extended Vasicek Model

- Specified under P by

$$\Lambda := \begin{pmatrix} \lambda_{11} & 0 & 0 \\ \lambda_{21} & \lambda_{22} & 0 \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

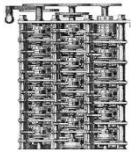
$$\Theta := \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$$

$$\Sigma := \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$r(t) := \delta_0 + \delta_1 y_1(t) + \delta_2 y_2(t) + \delta_3 y_3(t)$$

- This Dai & Singleton $A_0(3)$ model with **16 parameters** is not identified under P unless $\Theta := 0$ which is only appropriate to Q and has other difficulties



Economic Factor Model

- A 3 factor extended Vasicek Gaussian model specified under the market measure P by

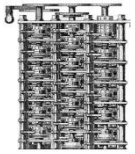
$$d\boldsymbol{X}(t) = (\mu_X - \lambda_X X(t) + \gamma_X \sigma_X)dt + \sum_{j=1}^{3} \sigma_{1j} d\boldsymbol{W}_j(t) \qquad \text{Long rate}$$

$$d\boldsymbol{Y}(t) = (\mu_Y - \lambda_Y Y(t) + \gamma_Y \sigma_Y)dt + \sum_{j=1}^{3} \sigma_{2j} d\boldsymbol{W}_j(t) \qquad \text{Minus Slope}$$

$$d\boldsymbol{R}(t) = \{k[X(t) + Y(t) - R(t)] + \gamma_R \sigma_R\}dt + \sum_{j=1}^{3} \sigma_{3j} d\boldsymbol{W}_j(t)$$

Unobservable instantaneous short rate

- Its discretization is estimated from CMS swap data with many observed yield curve points – rates – from 1 day (Libor) to 30 years (Treasury) using the EM algorithm which iterates Kalman filtering and maximum likelihood estimation to convergence
- Specifying the constant market prices of risk in terms of volatility units solves the ***X*** & ***Y*** identification problem and setting them to zero generates the factor pricing process
- This workhorse model has been used for pricing complex products and ALM using daily to quarterly frequency data in US, UK , EU, Swiss and Japanese jurisdictions



State Space Model Formulation

Transition Equation

$$\boldsymbol{Y}_t = d + \Phi Y_{t-1} + \boldsymbol{\eta}_t,$$

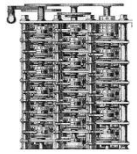
$$E[\boldsymbol{Y}_t | Y_{t-1}] = d + \Phi Y_{t-1}$$

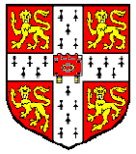
$$var(\boldsymbol{\eta}_t) = var(\boldsymbol{Y}_t | Y_{t-1}) = \Omega(Y_{t-1}) := \Omega_t$$

Measurement Equation

Substantial number of observed yields (e.g. 14)

$$y_t = \mathcal{A} + \mathcal{B} Y_t + \varepsilon_t$$



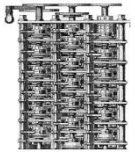


Calibrating the EFM Model

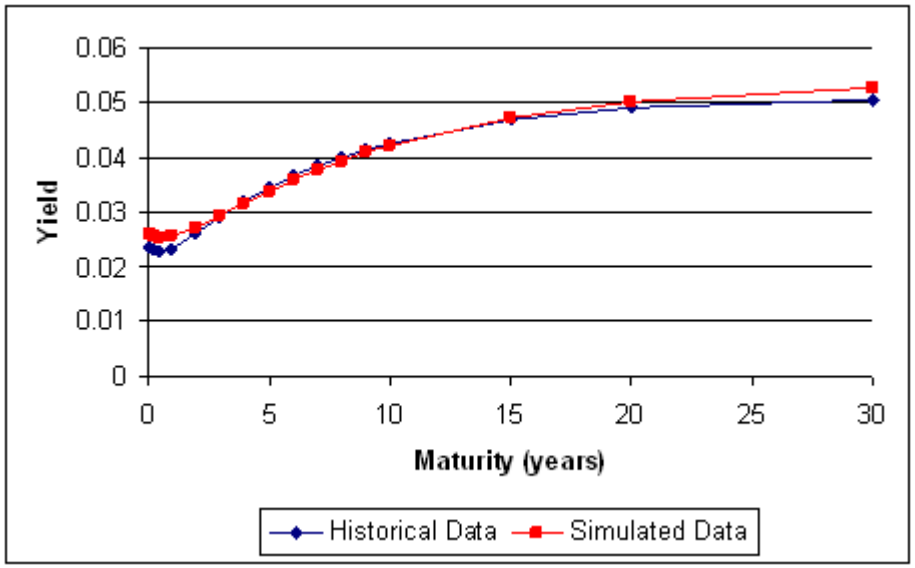
- Given the vector of parameters θ this Gaussian extended Vasicek model has rates (zero coupon bond yields) for maturity $\tau := T - t$ of the form

$$y(t,T) = \tau^{-1}[A(\tau,\theta)R_t + B(\tau,\theta)X_t + C(\tau,\theta)Y_t + D(\tau,\theta)]$$

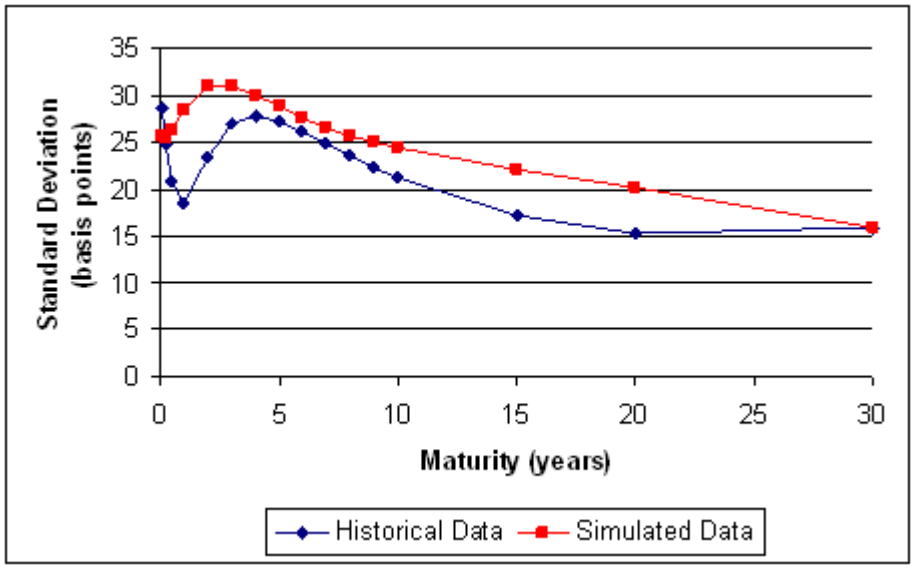
- We interpolate the appropriate swap curve linearly to obtain swap rates at all maturities and then use 1, 3 and 6 month LIBOR rates and the swap curve to recursively back out a zero coupon bond yield curve for each day from the basic swap pricing equation Ron (2000)
- This gives the input data for model calibration to give the parameter estimates $\hat{\theta}$
- Calibration is accomplished using the EM algorithm which iterates successively the Kalman filter (KF) and maximum likelihood estimation from an initial estimate θ_0
- At each iteration multi-extremal likelihood optimization in θ is accomplished using a global optimization technique followed by an approximate conjugate direction search
- The procedure is run on a Dell 48 Intel core system using parallelization techniques and we have also investigated the use of cloud computing for these calculations



EFM Model EU Yield Curve Prediction 2003



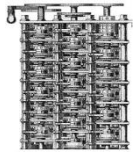
Mean level of yields over 2003 for historical and simulated weekly data



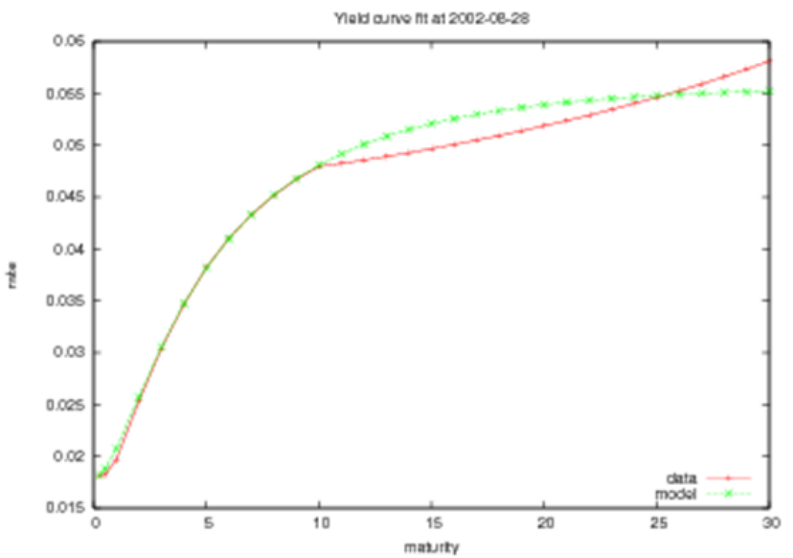
Weekly standard deviation of yields over 2003 for historical and simulated data

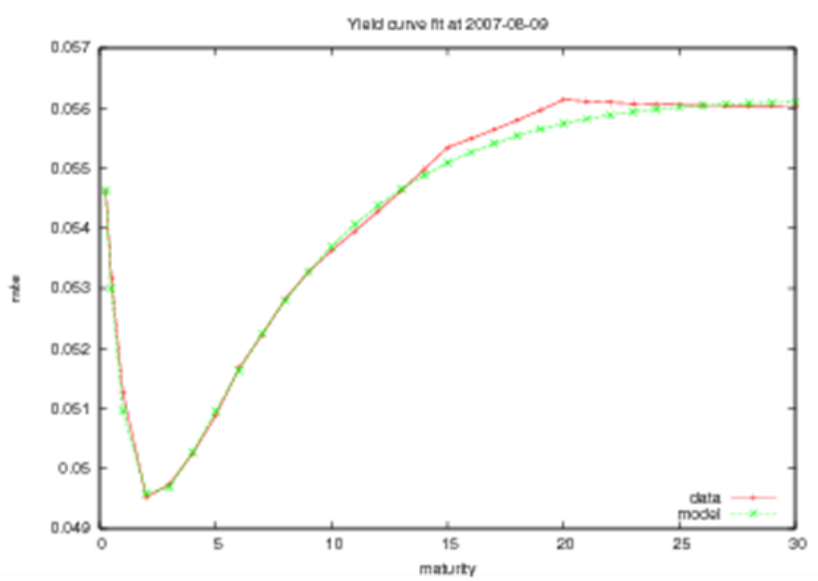
Dempster, Medova & Villaverde (2010)

- Longer term out-of-sample yield curve prediction has recently been independently found to be superior to the arbitrage-free Nelson-Siegel model of Christensen, Diebold & Rudebusch (2011) widely used by central banks



Goodness of Fit to Historical Yield Curves





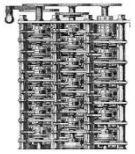
Current Environment

- In all the world's major economies low interest rates have prevailed since the 2007-2008 financial crisis which were presaged by more than a decade in Japan

- This has posed a problem for the widespread use of diffusion based yield curve models for derivative and other structured product pricing and for forward rate simulation for systematic investment and asset liability management

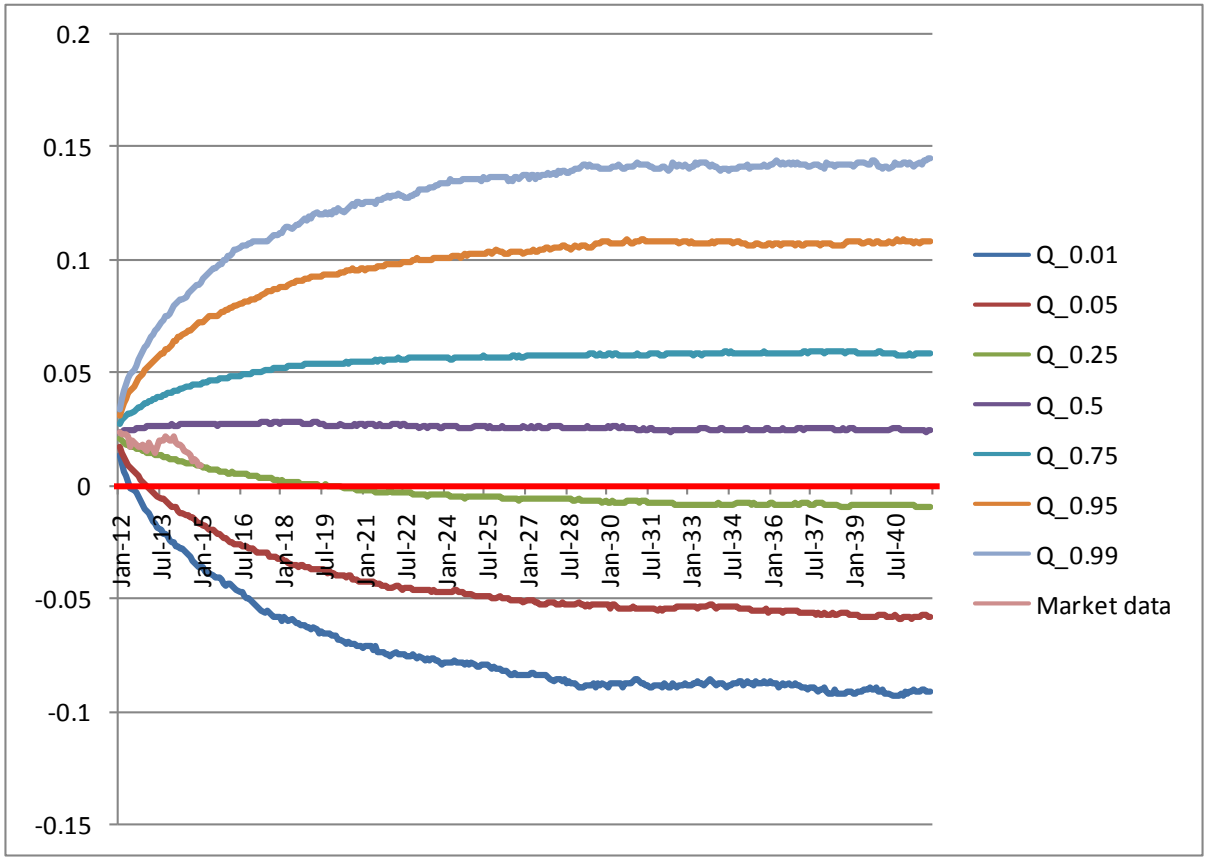
- Sufficiently accurate for pricing and discounting in relatively high rate environments Gaussian models tend to produce an unacceptable proportion of negative forward rates at all maturities with Monte Carlo scenario simulation from initial conditions in low rate economies

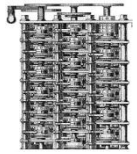
- The implications for this question of negative nominal rates in deflationary regimes and the currently fashionable multi-curve models remain to be seen



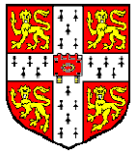
EFM Model Euro 10 Year Rate for 30 Years

Quantiles based on 100,000 scenarios





www.cambridge-systems.com

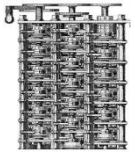


High Performance Computing Requirement

- Beginning with work in the Bank of Japan in the early 2000s there is currently considerable research in universities, central banks and financial services firms to develop yield curve models whose simulation produces nonnegative rate scenarios

- All this work is based on a suggestion of Fisher Black (1995) published posthumously to apply a call option payoff with zero strike to the model instantaneous short rate which leads to a piecewise nonlinearity in standard Gaussian affine yield curve model formulae for zero coupon (discount) bond prices and the corresponding yields and precludes their explicit closed form solution

- As a result most of the published solutions to Black-corrected yield curve models are approximations and even these require high performance computing techniques for numerical solution but we shall study here an obvious approximation which works extremely well as we shall see and is amenable to cloud computing for speed up



Nonlinear 3-Factor Black Model

- In a posthumously published paper Fisher Black (1995) suggested correcting *a priori* a Gaussian short rate model for a shadow short rate $\boldsymbol{r}$ to give the actual short rate as

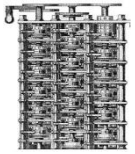
$$\boldsymbol{r}_{actual,t} := \max[0, \boldsymbol{r}_{shadow,t}] := 0 \vee \boldsymbol{r}_{shadow,t}$$

- Applied to an affine 3-factor Gaussian yield curve model such as that of JSZ or our EFM model this yields a hard nonlinear estimation problem posed by the bond price

Joslin, Singleton & Zhu (2011)

$$P_t(\tau) = E^Q \exp[-\int_t^{t+\tau} 0 \vee \boldsymbol{r}_{shadow,s}\, ds]$$

- Such models have been studied in the 2-factor case by the Bank of Japan and at Stanford but their discount bond pricing (rate) PDE methods do not easily extend to 3 factors Ichuie & Ueno (2007) Kim & Singleton (2011) Christensen & Rudebusch (2013) Kim & Priebsch (2013)

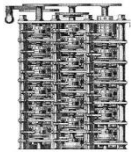


3-Factor Black Model Stylized Properties

Stylized Fact Properties	Yield Curve Model					
	CIR	BDFS	Vasicek	JSZ/HW	JSZ/HW/BRW	Black
	$A_3(3)$	$A_3(3)$	$A_3(3)$	$A_1(3)$	$A_0(3)$	$A_0(3)$
Mean Reverting Rates	Yes	Yes	Yes	Yes	No	Yes
Nonnegative Rates	Yes	No	No	No	No	Yes
Stochastic Rate Volatility	Yes	Yes	No	No	No	Yes*
Closed Form Bond Prices	Yes	Yes	Yes	Yes	Yes	No
Replicates All Observed Curves	No	Yes	Yes	Yes	Y es	Yes
State Dependent Risk Premia	No	No	No	Yes	Yes	Yes
Good for Long Term Simulations	No	No	No	No	No	Yes
Slow Mean Reversion Under *Q*	No	No	No	No	No	Yes
+ve Rate/Volatility Correlation	No	No	No	No	No	Yes
Effective in Low Rate Regimes	No	No	No	No	No	Yes

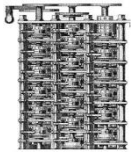
Table 1. Properties of evaluated yield curve models with regard to stylized facts

*Rate volatilities are piecewise constant punctuated by random jumps to 0 at rate 0 boundary hitting points.



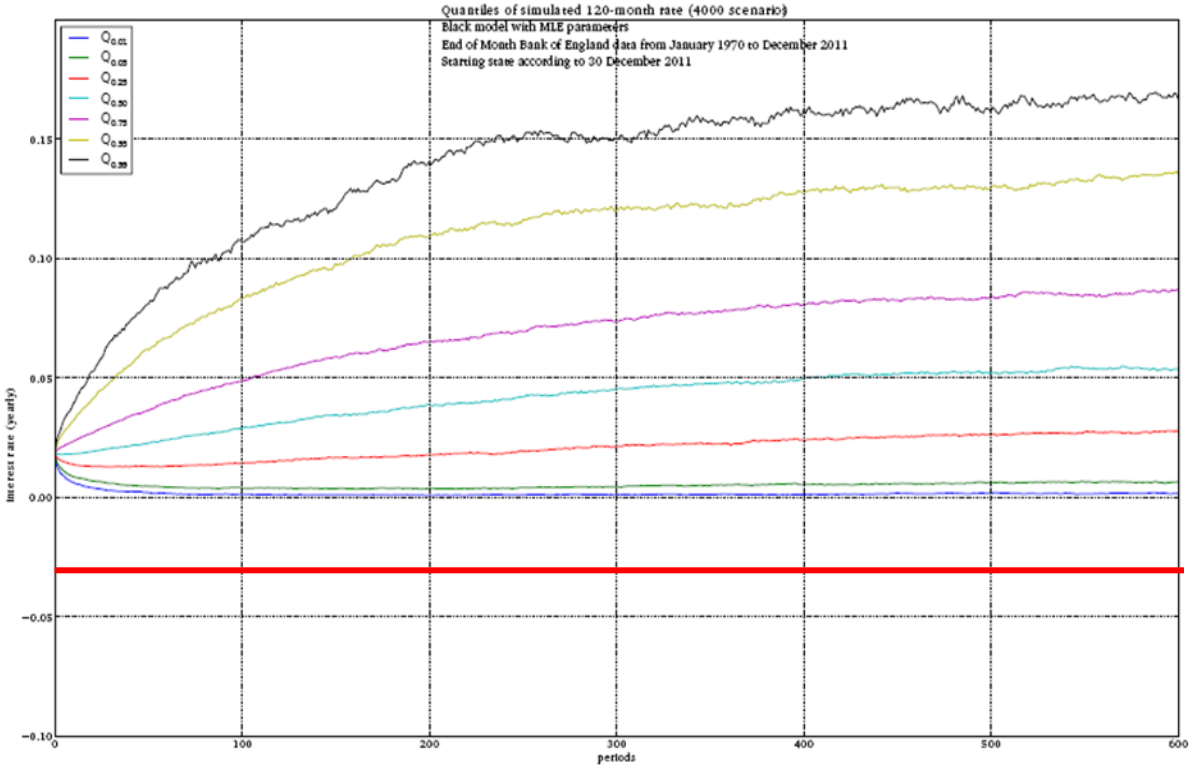
Black Model Calibration Progress

- Bonfim (2003) estimated his 2 factor model only on yields safely above 0 where the underlying shadow rate affine model rates and the Black rates agree, used the standard KF in the EM algorithm and solved the 2D parabolic quasilinear bond price PDE with finite differences
- Bauer & Rudebusch (2014) took the same approach to the 3 factor model employing the EKF in the EM algorithm and evaluated bond prices using 500 path Monte Carlo simulation as do Lemke & Vladu (2014)
- Dempster *et al.* (2014) used least squares with 4 observed yields, QMLE and analytical approximation for short yields and 10,000 path Monte Carlo for longer maturity yields as noted above
- We apply the Black correction to the measurement equation for yields within the unscented Kalman filter together with QMLE in the EM algorithm and EFM bond prices



Black Model 10 Year Gilt Rate
50 Year Predicted Distribution 2011-2061

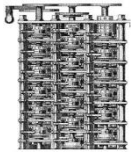
Quantiles based on 10,000 scenarios



Source: Dempster *et al.* (2014)

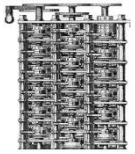
3 Factor Black Model Approaches

- The differences between current approaches to Black models based on 3 factor affine shadow rate models may be categorized in terms of handling the three steps crucial to the solution process
- Method of inferring (3 factor) states from observed market rates
  - inverse mapping or least squares
  - extended or iterated extended Kalman filter (EKF or IEKF) with piecewise linearization
  - unscented Kalman filter (UKF) with averaged multiple displaced KF paths
- Method of parameter estimation
  - method of moments
  - maximum likelihood (MLE) or quasi maximum likelihood (QMLE)
- Method of calculating bond prices or yields
  - Monte Carlo simulation
  - PDE solution
  - approximation



Monte Carlo Bond Pricing

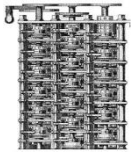
- Calibration of the nonlinear Black model with any underlying 3 factor Gaussian shadow rate model is **more** computationally intensive than for the underlying affine model
- Dempster, Evans & Medova (2014) use cloud facilities and Monte Carlo simulation with a JSZ 4 yield curve point model
- In more detail:
  - For short rates the closed form numerical rate calculations of Kim & Singleton (2011) are used
  - For long rates the averages of Monte Carlo forward simulated paths -- which automatically take account of the convexity adjustment otherwise required for this model – are used
- With this approach filtering a multi-curve EFM model for OTC structured derivative valuation becomes **very** computationally intensive



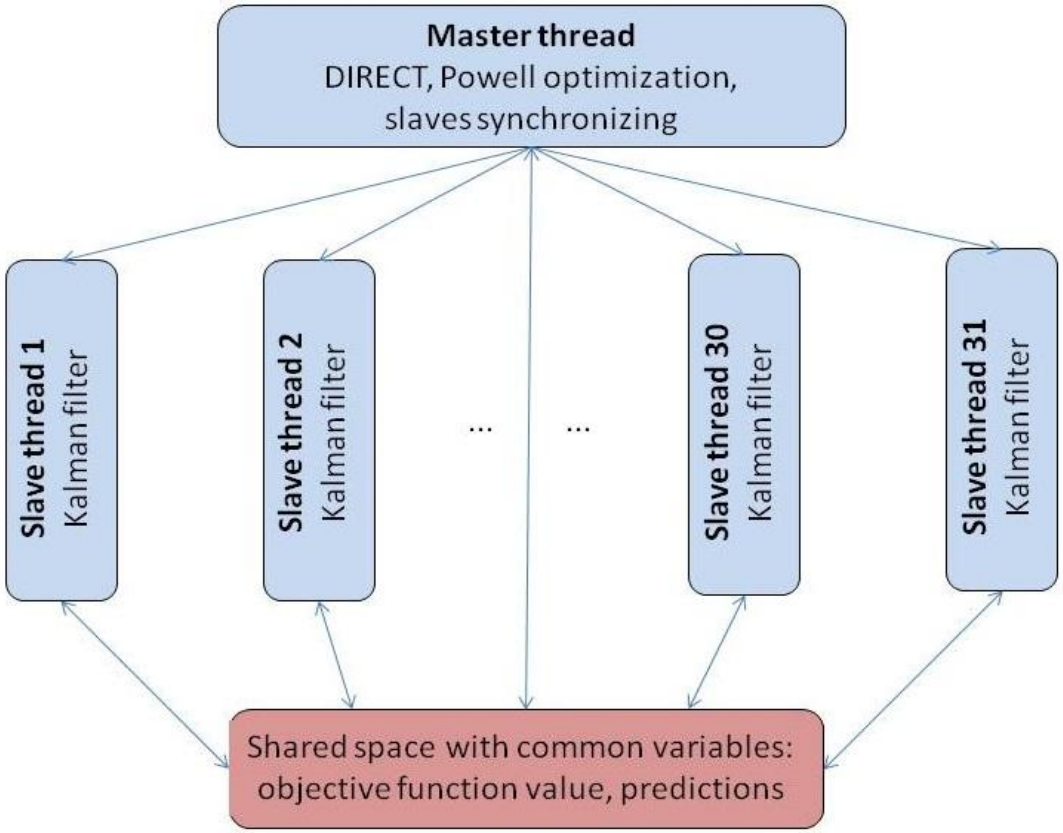
Unscented Kalman Filter Bond Pricing

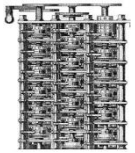
- Here we calibrate the Black EFM model with our current EM algorithm approach using the (NAG) unscented Kalman filter to handle the "hockey stick" nonlinearity

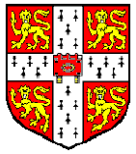
  Julier & Uhlmann (1997)
- Working with yields directly as we do rather than bond prices computed or approximated numerically from integrals of the instantaneous short rate as in the references to Black model calibration previously cited significantly accelerates computation
- Putting the EFM 3-factor yield curve dynamics in state-space form shows that the factor state dynamics remain linear Gaussian while the Black nonlinearity may be directly applied to each observed maturity market rate in the shadow rate affine measurement equation – longer maturity yields typically need no correction
- With this approach the 35 (34 sigma points plus original) duplicate KF calculations of the unscented Kalman filter averaged at each daily time step can be mindlessly parallelized to handle the Black nonlinearity in essentially the same running time as the calibration of the underlying EFM model using basic linear Kalman filtering



Parallelization Schema with MPI







Data

- Combination of LIBOR data and fixed interest rate swap rates (the ISDA fix) for each of 4 currency areas (EUR, GBP, USD, JPY) to bootstrap the yield curve daily for 14 maturities:

  3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years,

  6 years, 7 years, 8 years, 9 years, 10 years, 20 years, 30 years

- In the case of the Swiss franc (CHF), only 12 maturities are available:

  3 month, 6 month, 1 year, 2 years, 3 years, 4 years, 5 years,
  6 years, 7 years, 8 years, 9 years, 10 years

- Calibration periods used for these 5 currencies are the following:

  EUR: 02.01.2001 to 02.01.2012

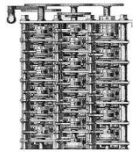
  CHF: 02.01.2001 to 31.05.2013

  GBP: 07.10.2008 to 31.05.2013

  USD: 02.01.2001 to 31.05.2013

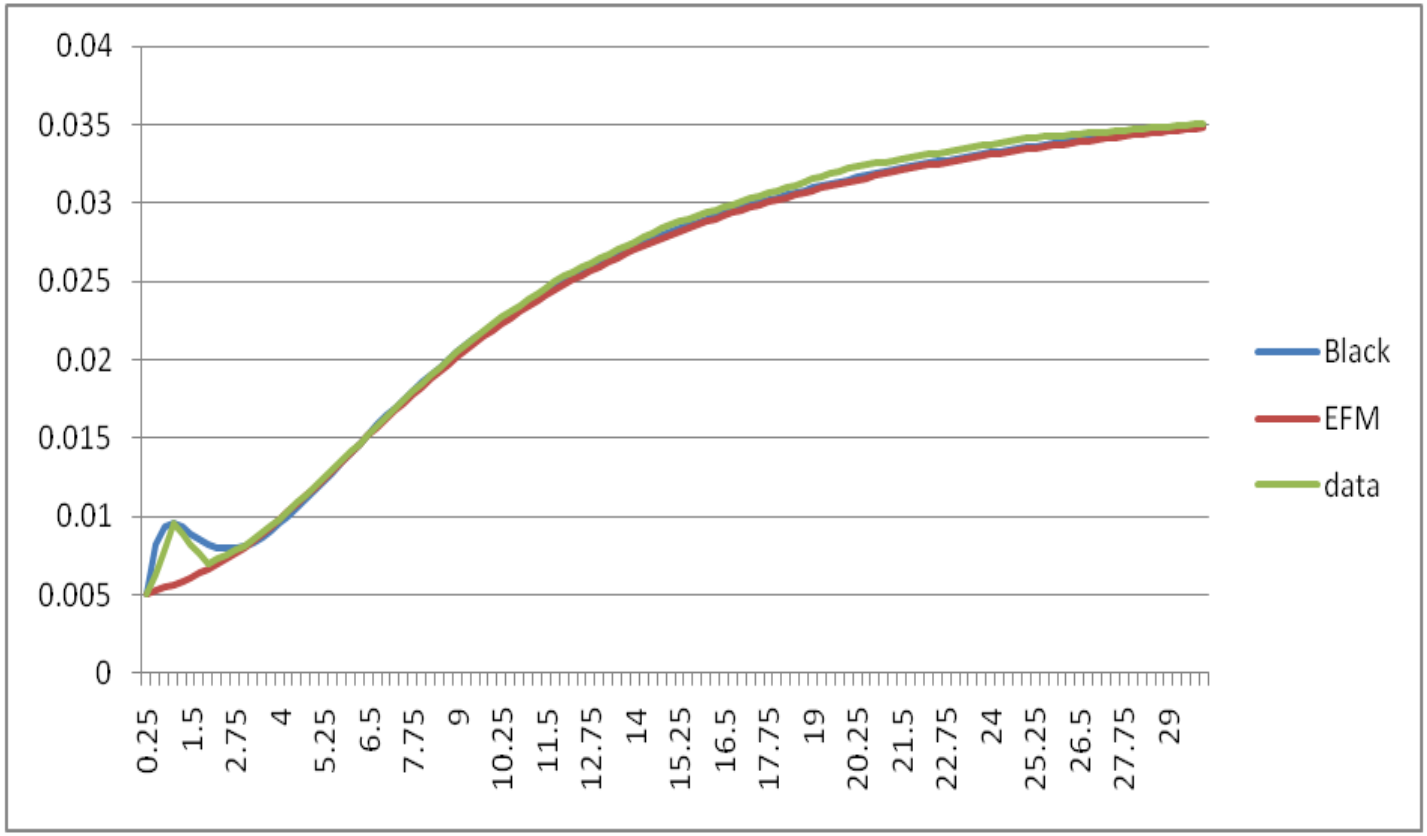
  JPY: 30.03.2009 to 31.05.2013

- The data was obtained from Bloomberg



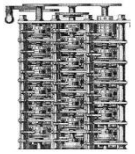
GBP **Date: 18 Feb 2013**

Model	RMSE
EFM	8 bp
Black EFM	5 bp



Overall In-sample Goodness of Fit

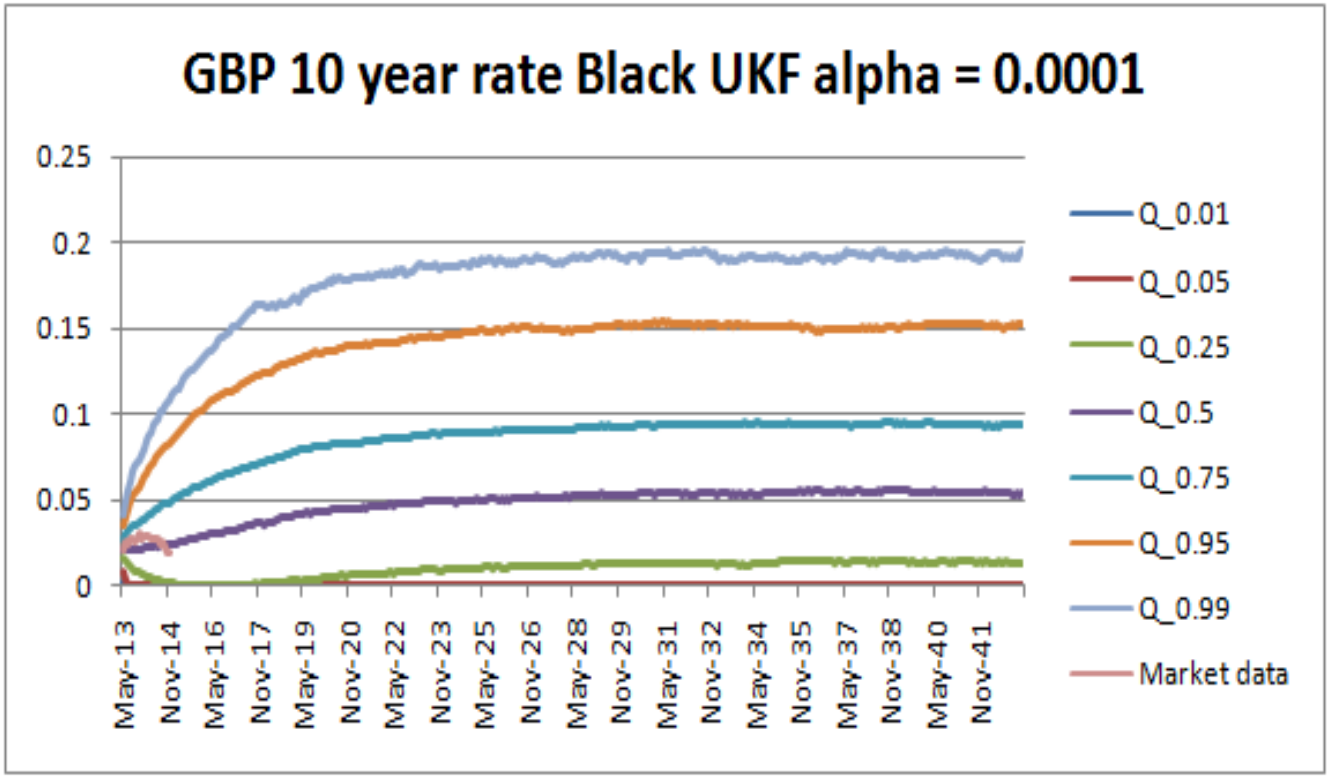
Currency	Observations	Calibration	log likelihood	Sample fit RMSE (vol)
EUR	2817	EFM	232,652	15 bp
		EFM UKF	252,500	17 bp
		Black EFM α:=0.0025	259,436	15 bp
CHF	3100	EFM	232,100	8 bp
		EFM UKF	250,391	10 bp
		Black EFM α:=1.0	253,095	8 bp
GBP	1171	EFM	98,021	16 bp
		EFM UKF	103,529	15 bp
		Black EFM α:=0.0001	105,368	14 bp
USD	3093	EFM	279,114	15 bp
		EFM UKF	280,745	25 bp
		Black EFM α:=0.001	292,954	22 bp
JPY	950	EFM	91,014	6 bp
		EFM UKF	84,564	28 bp
		Black EFM α:=0.006	102,544	6 bp



Monte Carlo Out-of-sample Projection

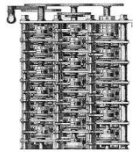
Quantiles based on 100,000 scenarios

30 Year Black EFM GBP 10 Year Rate

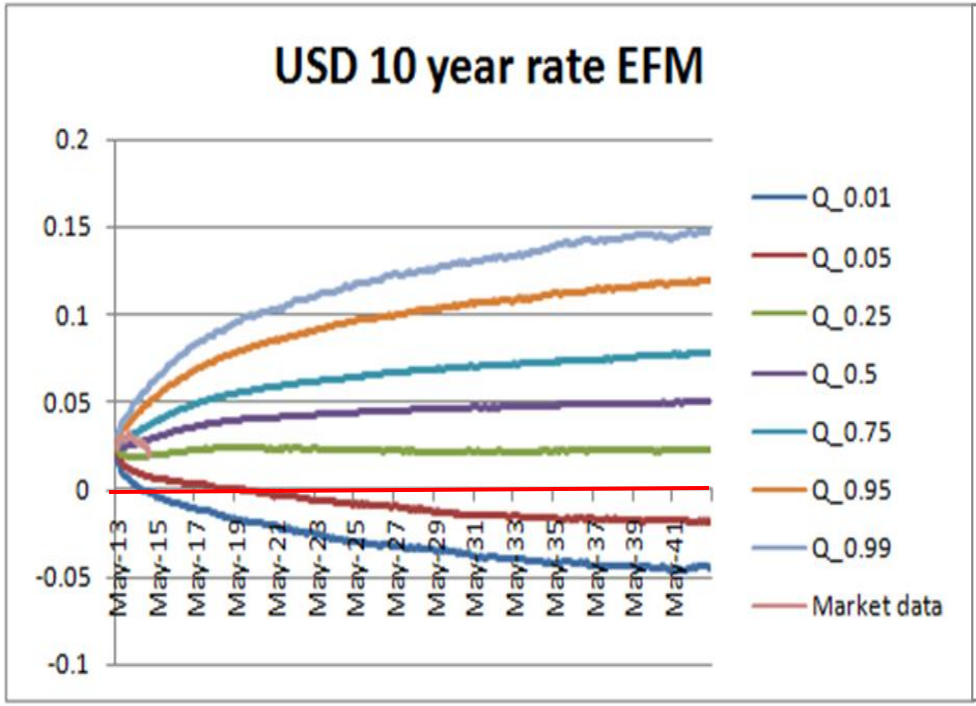


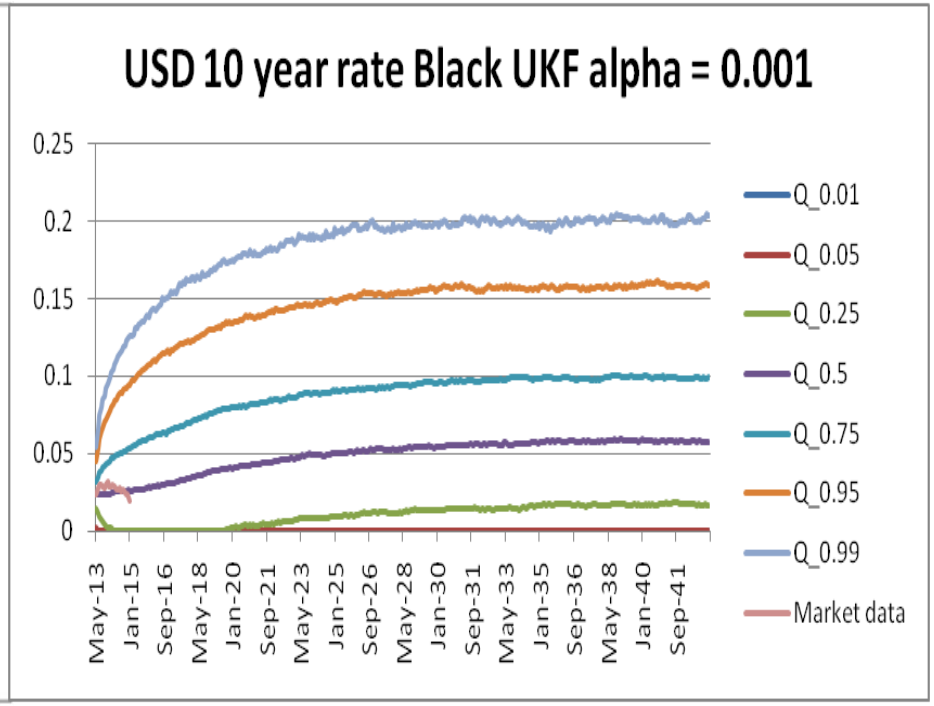
GBP 10 year rate forecast RMSE over 20 months

Black median	0.48%
EFM median	0.45%



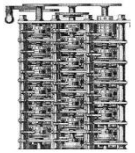
USD 10 Year Rate Out-of-sample Projections





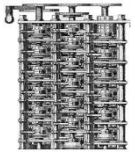
USD 10 year rate forecast RMSE over 21 months

Black median	**0.39%**
EFM median	**0.43%**

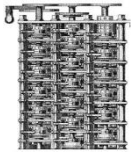


Findings

- We have developed a Black-corrected version of our workhorse 3 factor affine Gaussian yield curve Economic Factor Model implemented using the unscented Kalman filter to handle the Black nonlinearity and HPC techniques
- Although this method generates an approximation to the full Black model its accuracy is comparable to and its computing run time only about twice that of the basic EFM model – unlike all the alternatives published to date which are very heavily computationally intensive
- Using the NAG UKF algorithm (g13 ejc) with tuned α parameter setting both the in- and out-of-sample accuracy of the method exceeds that of the affine EFM model and it possesses much better dynamics
- Using the cloud we can reduce calibration times on big samples to minutes



Pricing Variable Annuity Products



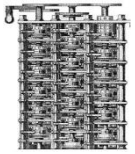
GMAB VA Portfolio Fund Risk Management

- Consider an illustrative problem in which after initial client cash outlays no GMAB contributions are allowed

- Liabilities: nominal or index-linked guarantees
  - Nominal guarantee: Fixed percentage of the initial wealth is guaranteed at a specified date
  - Inflation, equity – or other capital market index – linked guarantees

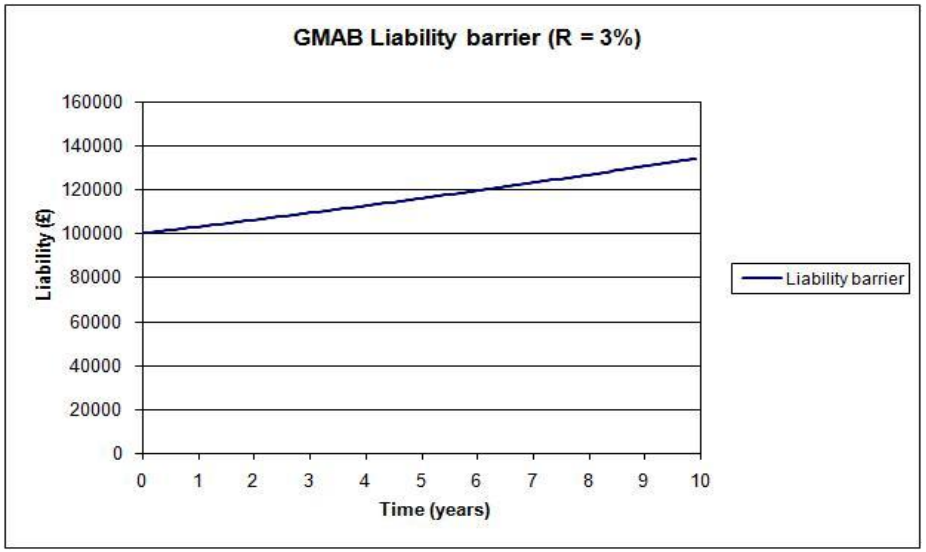
- Assets: EU bonds with maturity 1, 2, 3, 4, 5, 10 and 30 years and the Eurostoxx 50 index

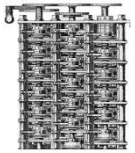
- Transaction costs: At annual bond rollovers



Barrier Formulation - GMAB

- A premium of £100,000 is received from the client at the outset
- The maturity of the product is 10 years
- The guaranteed rollup rate *R%* (per annum) is fixed throughout the planning horizon and compounded monthly





GMAB Annuity – 3% Guarantee

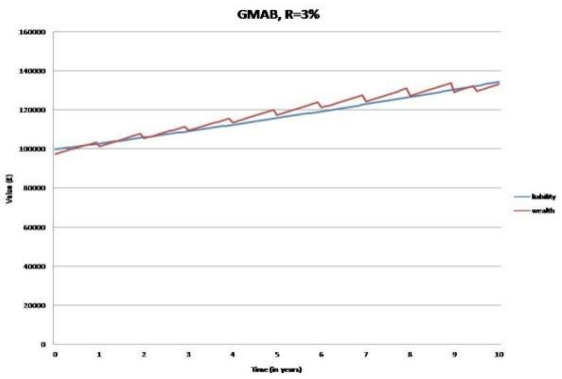
The initial portfolio allocation is comprised entirely of AA bonds with 1-year maturity – a slightly more risky strategy due to AA credit risk

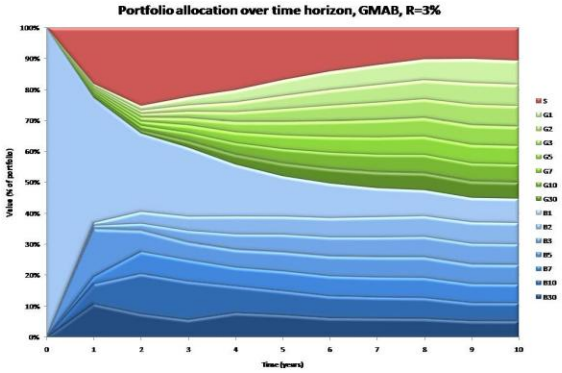
Investment in equity and AA bonds is preferred between years 1-3

Portfolio diversification occurs between years 3-5

Conservative strategy after 5 years

Matching the 3% barrier is realistically possible





Barrier Formulation - GMIB

The product is split into 3 phases

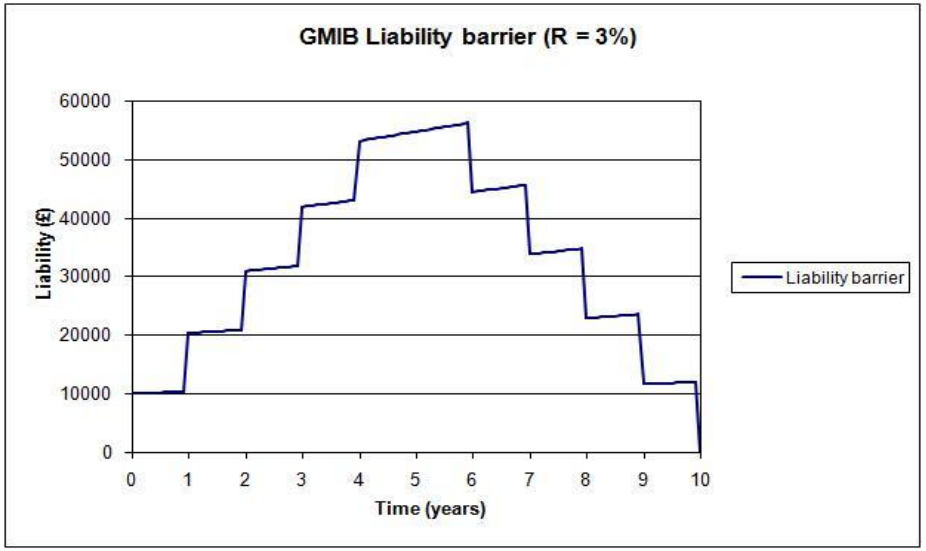
- 4-year accumulation phase
- 2-year election period (fixed)
- 4-year distribution phase

A contribution of £10,000 is received from the policyholder on the first trading day of each year during the 4-year accumulation phase (**5** policyholder **contributions** in total)

The total growth phase of the annuity is **6 years**

The guaranteed rollup rate *R%* (per annum) is fixed throughout the planning horizon and compounded monthly

Guaranteed income payments are delivered to the policyholder during the 4-year distribution phase of the annuity. There are a total of **5** income **payments** to the policyholder



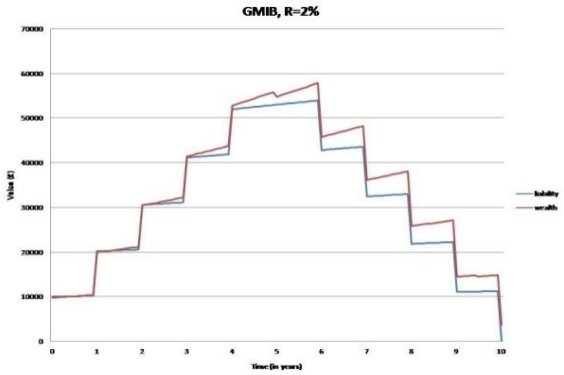
GMIB Annuity – 2% Guarantee

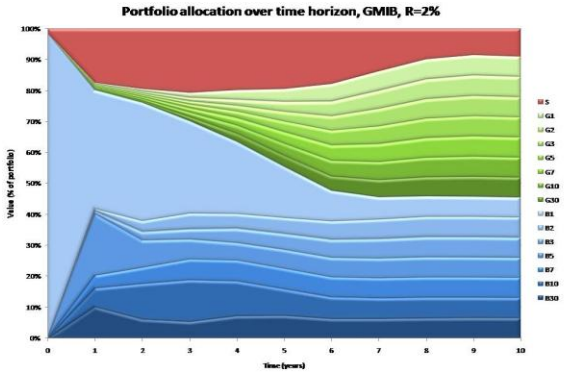
Initial portfolio comprises entirely of 1-year AA bonds

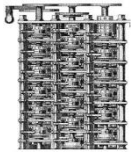
Steady diversification over years 1-6 with good growth during years 4-6

Equity holding never exceeds 20%

Expected terminal profit of £3,640 net of charges amounting to a further £8,320 relative to the £50,000 collected and dispersed over the ten year horizon







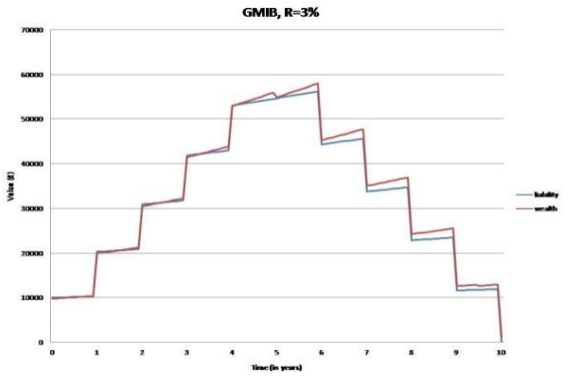
GMIB Annuity – 3% Guarantee

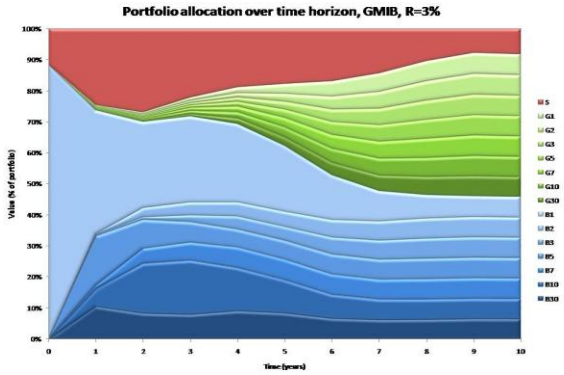
Initial portfolio comprises equity (12%) and 1-year AA bonds (88%)

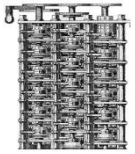
Higher equity holding during years 1-4 to track the liability barrier

Steady diversification during years 4-6

Expected terminal profit of £1,030 net of charges amounting to a further £8,210 relative to the £50,000 collected and dispersed over the ten year horizon

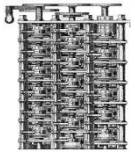




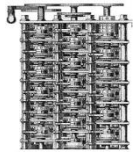


Other Variable Annuities

- Simple models for GMWB and GMDB
- More complex annuity products
  - Longer maturities
  - Variable interest rate structures
  - Complex phase structures
- Barriers inflation-linked
- Incorporation of other factors such as policyholder age and dynamic/base lapses
- All complex products can be priced using this approach

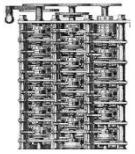


Unit Linked Guaranteed Product Fund Management



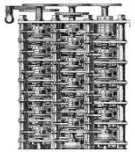
Unit Linked GMIB VA Fund Management

- Case study based on 10 year development of management of a family of funds backing a variety of guaranteed return open ended investment products and maturities provided in several jurisdictions by a major financial institution
- Guarantees were absolute and relative to various indices
- Over time investors were moved from one fund to another more appropriate to their current age – life staging
- Here we will look at simplified closed end model to show the power of the technology



Closed-End Guaranteed Return DC Fund

- After initial cash outlay no contributions are allowed
- Liabilities: nominal or index-linked guarantees
  - Nominal guarantee: Fixed percentage of the initial wealth is guaranteed at a specified date
  - Inflation- or other capital market index linked guarantees
- Assets: EU bonds with maturity 1, 2, 3, 4, 5, 10 and 30 years and the Eurostoxx 50 index
- Transactions costs: At annual bond rollovers
- At the decision times the zero coupon yield with maturity T is a proxy for the fixed coupon rate of a coupon-bearing bond with maturity T



ALM Formulation

- **Given a set of assets, a fixed planning horizon and a set of rebalance dates find the trading strategy that maximizes the risk-adjusted wealth and minimizes the shortfall below the PV of the guarantee subject to the constraints**

$$\max_{\left\{\begin{array}{l} portfolio\ \ bebalancing\ \ desicions: \\ a\in A,\omega\in\Omega,t\in T^d\cup\{T\} \end{array}\right\}} E\left\{\alpha\left(wealth\right)-\beta\left(shortfall\right)\right\}$$

subject to the specific **constraints**

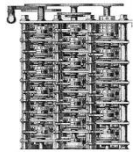
- **Scenario tree for the future assets returns**

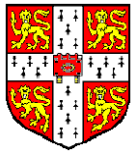
- **Liability barrier at any time t on scenario w**

$$L_t(\omega)=W_0(1+G)^T Z_t(\omega)=W_0(1+G)^T e^{-\mathbf{y}_{t,T-t}(\omega)(T-t)}$$

where G is an **annual guaranteed return**, $y_{t,T}$ is the **yield** of a **zero - coupon bond** which pays 1 at **maturity** T with **value** $Z_t(\omega)$ at time t on **scenario** ω

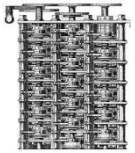
- **Economic three factor model is used for yield curve simulation**
  Medova *et al.* (2006) Dempster *et al.* (2010) Dempster *et al.* (2015)



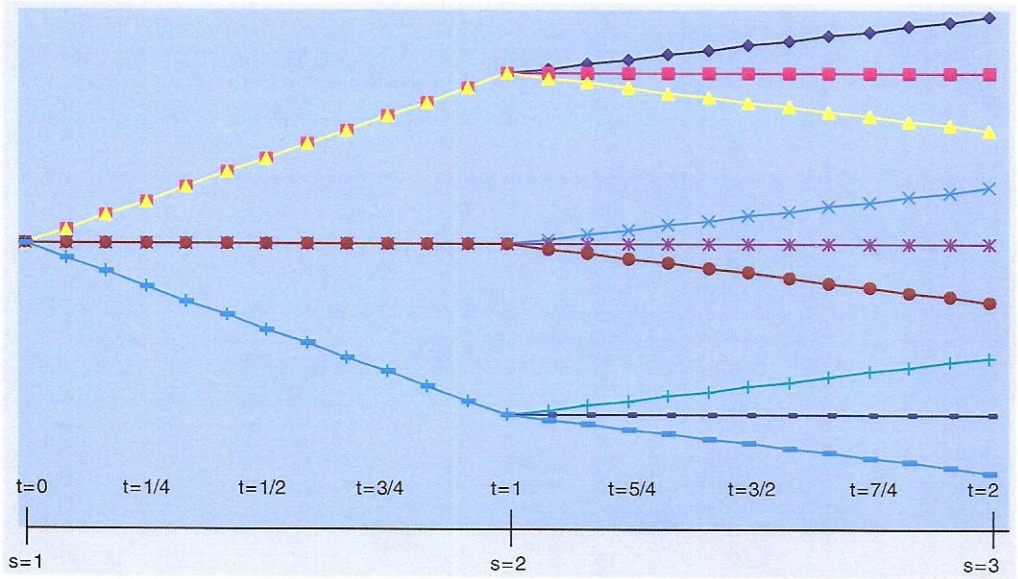


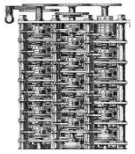
Scenario Generation

- Data : monthly (July 1997 – January 2003)
- Maximum likelihood estimation of the parameters of the stochastic process for Eurostoxx 50
- Recursive Kalman filter/ML estimation of the parameters for the yield curve
- Historical contemporaneous cross correlations
- Simulation of the conditional scenarios for Eurostoxx 50 and EU treasury yields

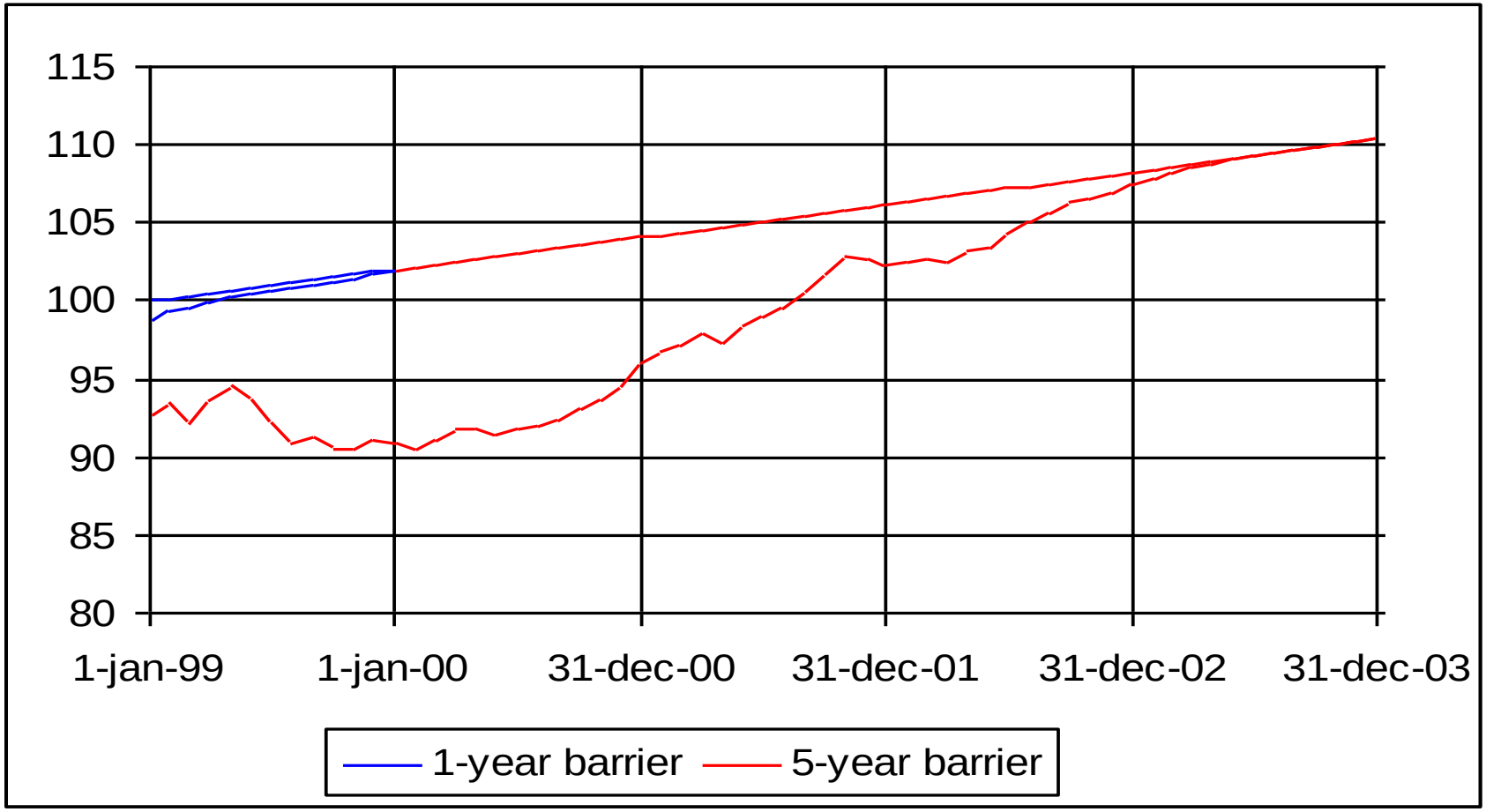


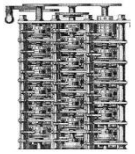
Graphical Representation of Scenarios

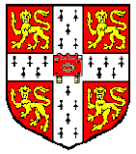




Liability Barrier for Long/Short Term Funds

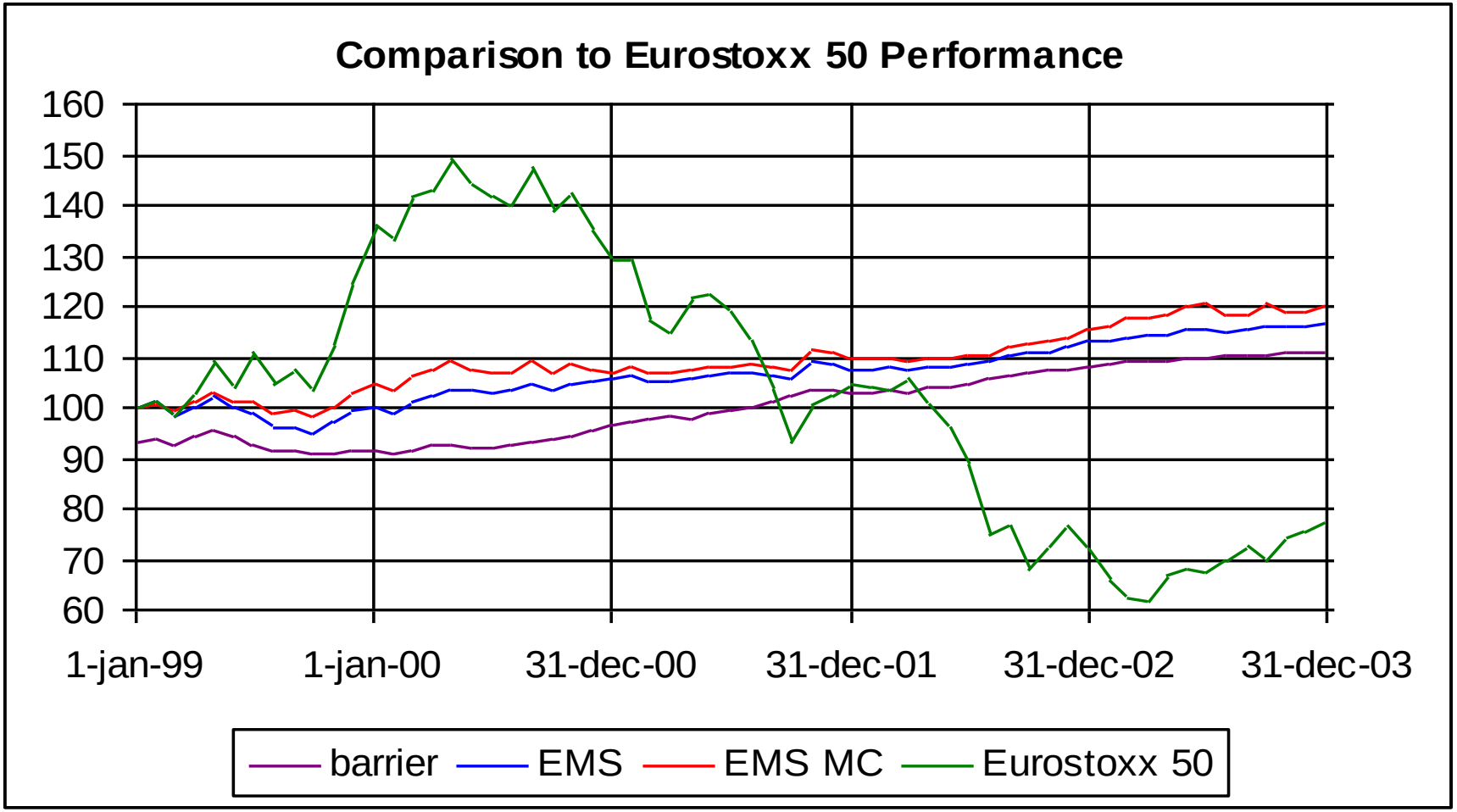






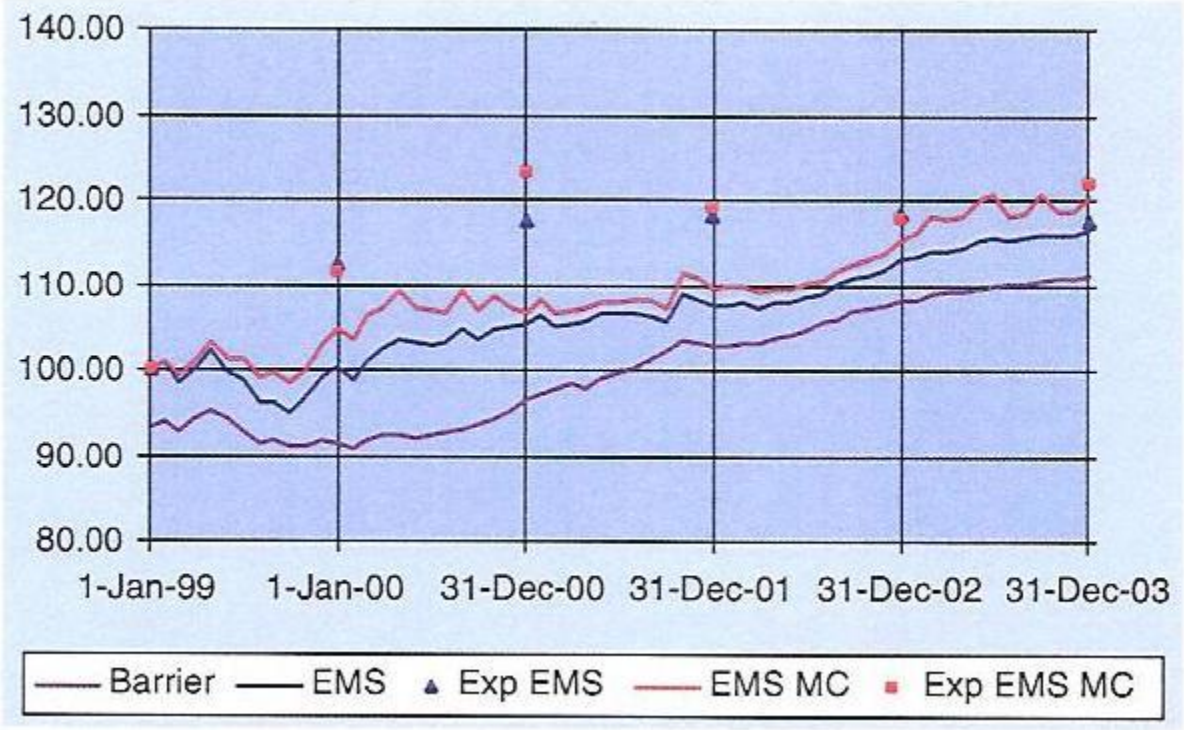
Historical Backtest 1999-2004

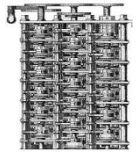
Expected maximum shortfall with monthly checking using the 512.2.2.2 Tree



Model Predictions and Historical Performance

Expected maximum shortfall for the 512.2.2.2.2 tree



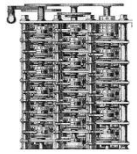


Portfolio Allocation

Expected maximum shortfall with monthly checking using the 512.2.2.2.2 tree

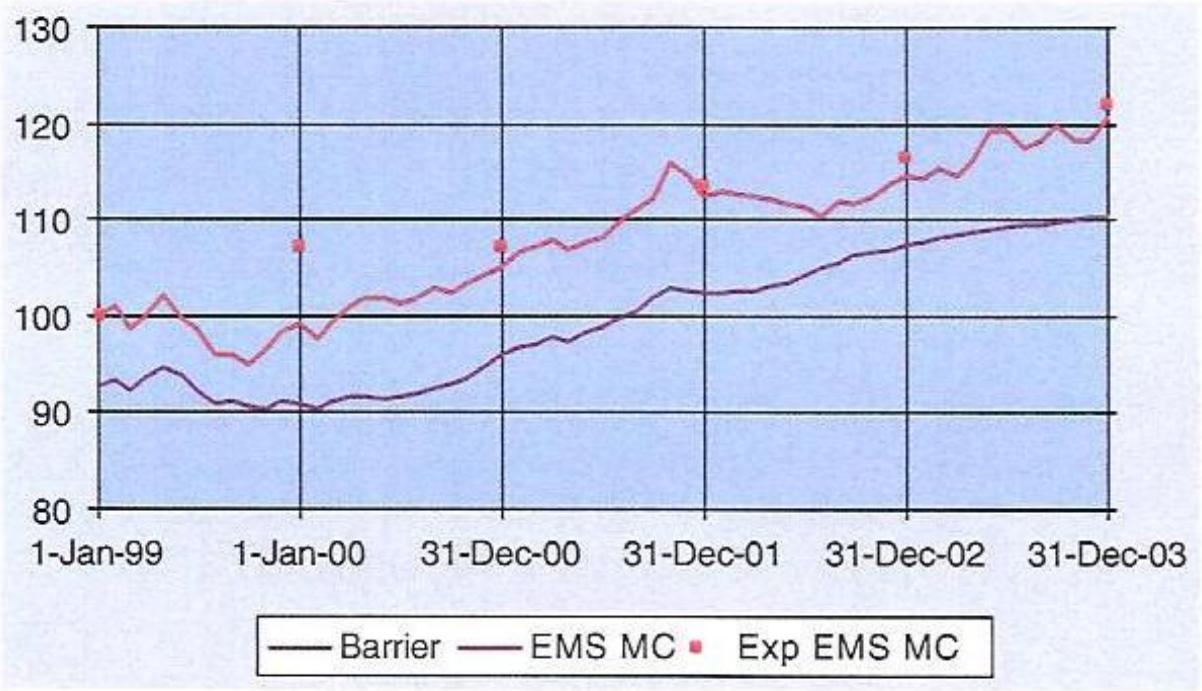
	1y	2y	3y	4y	5y	10y	30y	Stock
Jan 99	0	0	0	0	**0.69**	**0.13**	0	**0.18**
Jan 00	0	0	0	0	**0.63**	0	0	**0.37**
Jan 01	0	0	0	0	**0.37**	**0.44**	0	**0.19**
Jan 02	0	0	0	0	**0.90**	0	0	**0.10**
Jan 03	0	0	**0.05**	0	**0.94**	0	0	**0.01**

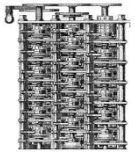
Longer bond maturities and smaller bond positions than other versions



GBM with Poisson Jumps Equity Index Process

Expected maximum shortfall with monthly checking using the 512.2.2.2.2 tree



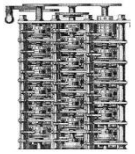


GBM with Poisson Jumps Equity Index Process Portfolio Allocation

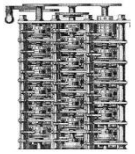
Expected maximum shortfall with monthly checking using the 512.2.2.2.2 tree

	1y	2y	3y	4y	5y	10y	30y	Stock
Jan 99	0	0	0	0	**0.12**	**0.77**	0	**0.11**
Jan 00	0	0	0	0	0	**0.86**	0	**0.14**
Jan 01	0	0	0	0	**0.43**	**0.56**	0	**0.01**
Jan 02	0	0	0	0	**0.70**	**0.11**	0	**0.19**
Jan 03	0	0	0	0	**0.04**	**0.81**	0	**0.15**

Solution takes advantage of decreased 2003 and 2004 volatility to increase stock holdings

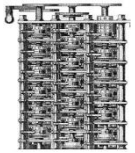


Conclusion



Conclusions

- Dynamic stochastic programming is the paradigm for asset liability management which is also applicable to individual household lifetime financial planning
- Ability to perform cash flow based optimal dynamic asset liability management over very long term random horizons in what-if mode
- Better idea of risks arising from future decisions – you can explicitly plan for them rather than adapting to outcomes as best you can as you go along myopically
- Demonstrably superior to current financial techniques in specific applications using sophisticated yield curve models



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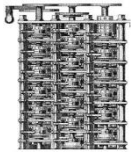

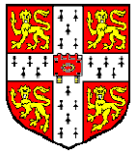

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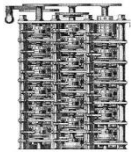

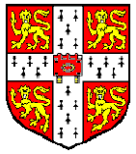

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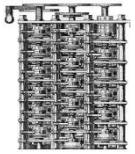

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